MINIMUM LEAF NUMBER OF 2-CONNECTED CUBIC GRAPHS

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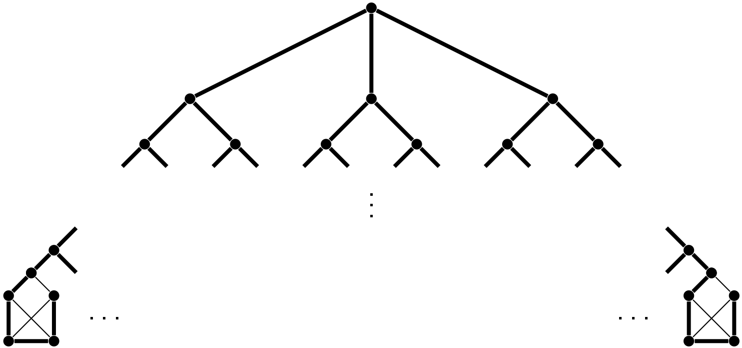

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Theorem (Goedgebeur, Ozeki, Van Cleemput, Wiener, 2018)

If G is a cubic graph of order n, then $m\ell(G) \leq \frac{n}{6} + \frac{1}{3}$.

- this bound is sharp, examples are not 2-connected

2-connected cubic graphs

Theorem (Goedgebeur et al.)

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If G is a 2-connected cubic graph, then $m\ell(G) \leq \lceil \frac{n}{10} \rceil$.

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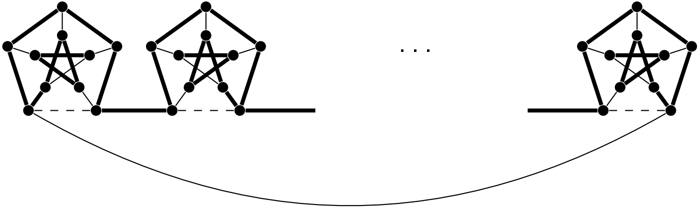

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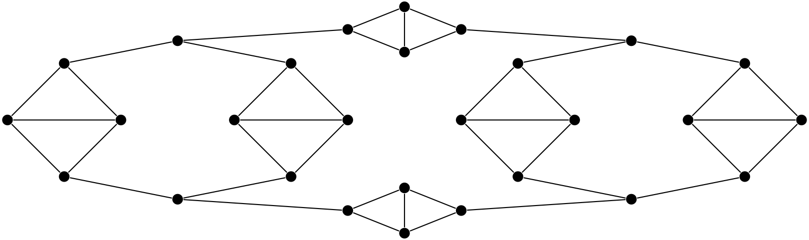

n=28, $m\ell(G) = 3$

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Theorem (DM, 2019+)

If G is a 2-connected cubic graph, then $m\ell(G) \leq \frac{n}{8} + 1$.

Sketch of the proof

Definition

(T, r) is called a *rooted tree* if T is a tree, and the *root* r is a vertex of T.

Definition

The *depth* of a vertex is its distance from the root, the depth of a rooted tree is the sum of the depths of its vertices.

Sketch of the proof

- Let T be a rooted spanning tree of G such that:
  1. it has the least possible number of leaves
  2. given the first condition, it has the largest possible depth.

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Proposition

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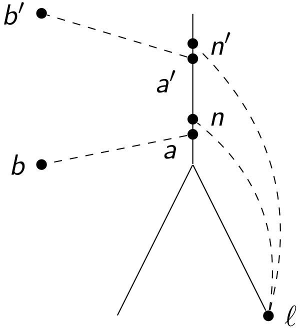
- $\ell - 1$ leaves, $6 \cdot (\ell - 1)$ vertices of degree 2 associated with the leaves, $\ell - 2$ vertices of degree 3

- $\ell + 6 \cdot (\ell - 1) + (\ell - 2) \leq n$, so $\ell \leq \frac{n}{8} + 1$.

Vertices of degree 2



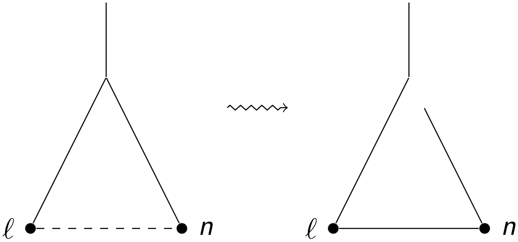
Case 1:

- There is a vertex of degree 3 on the path connecting the leaf ℓ and n / n' in T
- a and a' are not adjacent

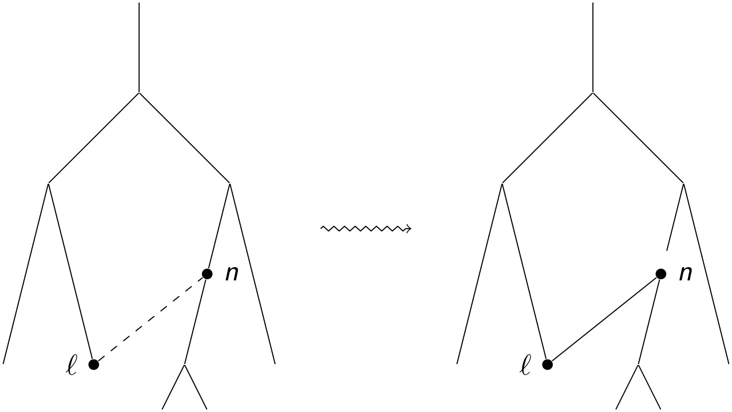
The vertices of degree 2 associated with ℓ are n, n', a, a', b and b'

Vertices of degree 2

- n and n' are not leaves:



- n and n' are ancestors of ℓ:



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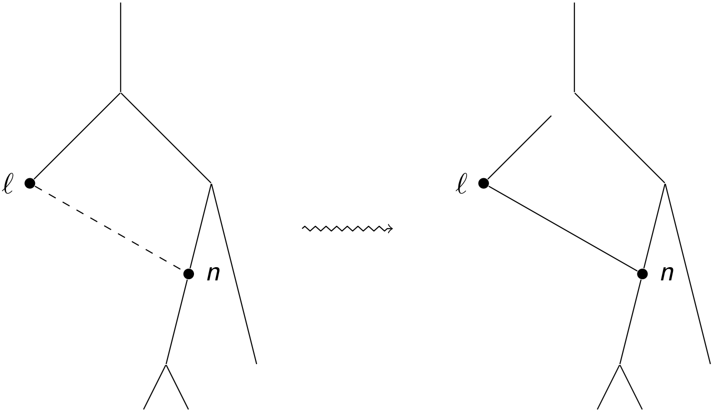
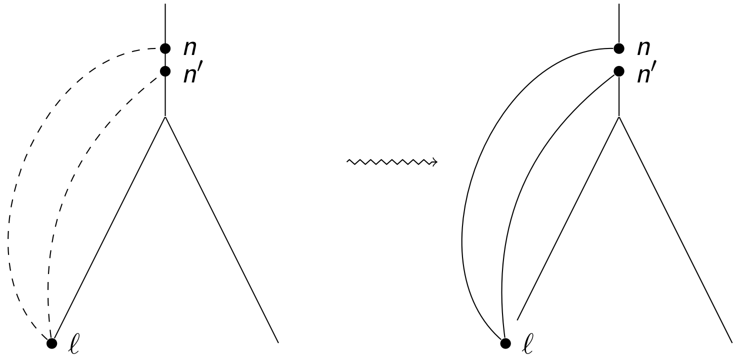
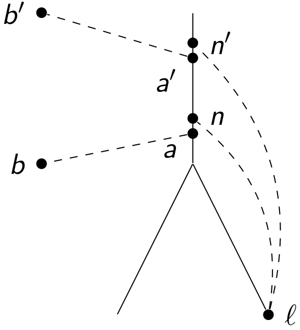


Vertices of degree 2

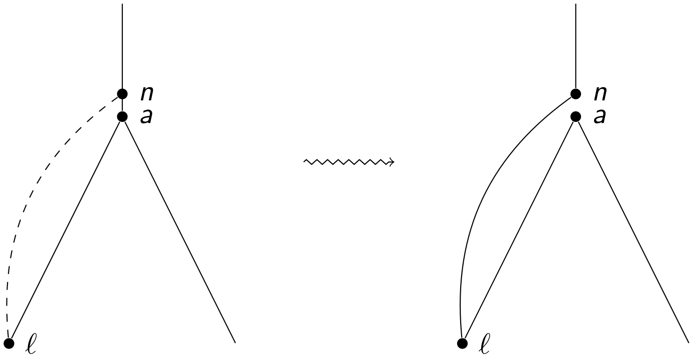
- n and n' are not a parent and a child:




b'
n'
a'
n
b
a
ℓ

Vertices of degree 2

- a and a' are vertices of degree 2:



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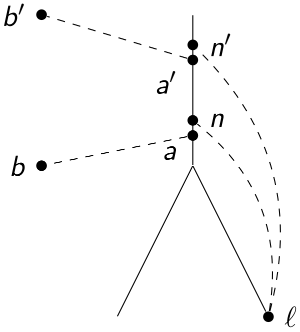
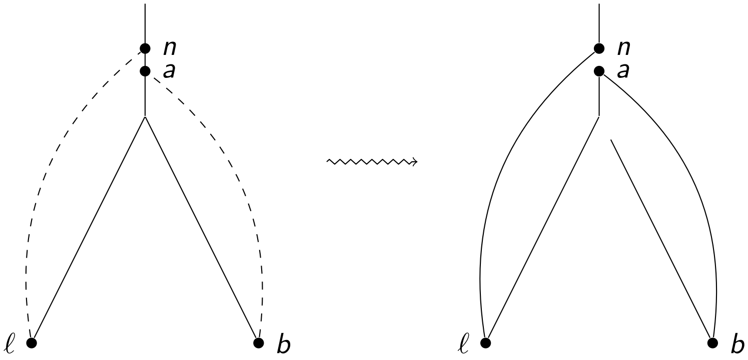
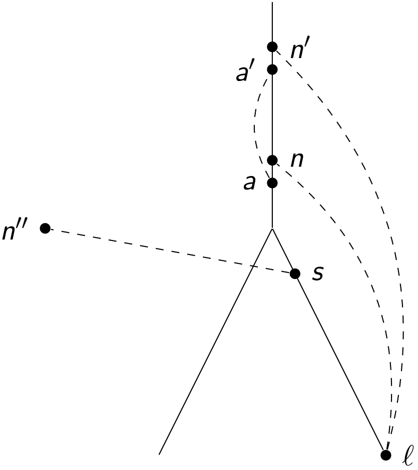


Vertices of degree 2

- b and b' are vertices of degree 2:



Other cases



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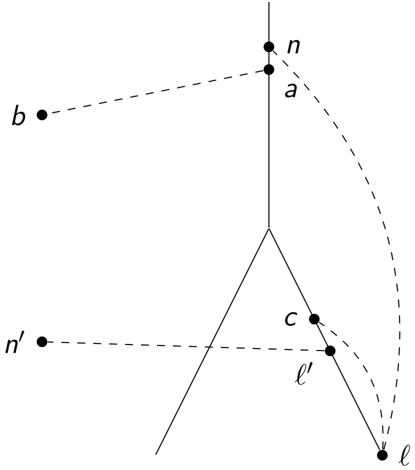

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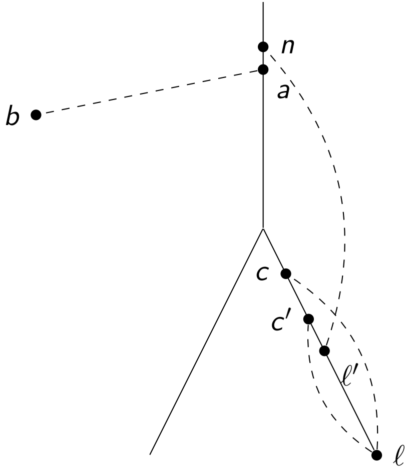

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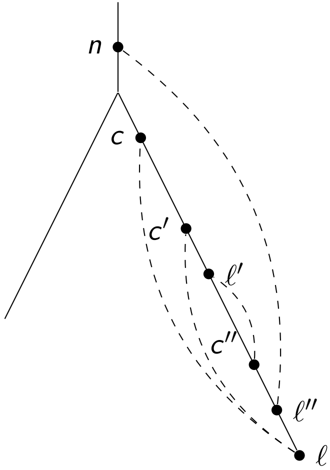

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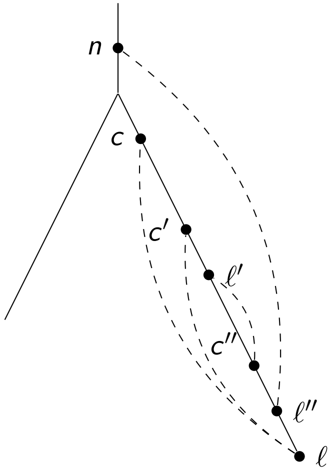

- This is the only case where the proof uses the 2-connectedness

Questions, related problems

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cubic 2-connected multigraphs - examples with $m\ell(G) = \frac{n}{6}$

Theorem (Boyd, Sitters, van der Ster, Stougie, 2011)

If G is a cubic multigraph with order n, then $m\ell(G) \leq \frac{n}{6} + \frac{2}{3}$.

Thank you!