Polynomial Chi-binding functions and forbidden induced subgraphs – a survey

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Chromatic Number

András Gyárfás (Budapest, 1985)

Problems from the world surrounding perfect graphs

Chromatic Number

Conjecture (Gyárfás)

Let T be any tree (or forest). Then there is a function f_T (χ-binding function) such that every T-free graph G satisfies

$$\chi(G) \leq f_T(\omega(G))$$

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Theorem (Erdös, 1959)

For every $k, g \geq 3$, there is a graph G with girth g and chromatic number k.

Coloring, sparseness, and girth, 2015
Alon, Kostochka, Reiniger, West, and Zhu

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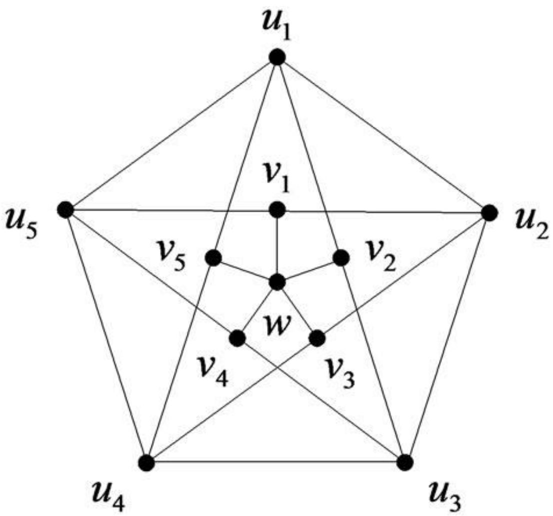
Mycielski construction

A sequence of graphs $G_2, G_3, \ldots$ with $\omega(G_k) = 2$ and $\chi(G_k) = k$.

$G_2 = K_2, G_3 = C_5, G_4$ is the Grötzsch/Mycielski graph



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Theorem (Gyárfás, 1987)
There is a chi-binding function for

- Stars
- Paths
- Brooms

- Trees with radius 2 (Kierstead and Penrice)
- Special trees with radius 3 (Kierstead and Zhu)
- pK2 (Wagon 1980)

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Theorem (Gyárfás, 1987)

If T is the path P_k then

$$\frac{R(\lceil k/2 \rceil, \omega + 1) - 1}{\lceil k/2 \rceil - 1} \leq \chi(G) \leq (k-1)^{\omega(G)-1}$$

Theorem (Esperet, Lemoine, Maffray, Morel, 2013)

If T is the path P_5 then

$$\chi(G) \leq 5 \cdot 3^{\omega(G)-3}$$

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Strong Perfect Graph Conjecture (Berge, 1960)
A graph G is perfect if and only if it does not contain an odd hole nor an odd antihole.

The SPGC became the
Strong Perfect Graph Theorem in 2002 by Chudnovsky, Robertson, Seymour, and Thomas.

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Problem (Gyárfás, 1987)

What is the order of magnitude of f_{P_5}?

Theorem (Gyárfás,1987)
There is no linear binding function for P5-free graphs.

$$\text{Known: Let H be a graph with } \alpha(H) \geq 3, \text{ then}$$

$$f(P_5, H) \geq c \cdot \frac{\omega(G)^2}{\log \omega}$$

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This paper of (Gyárfás) contains several other challenging conjectures.

Polynomial chi-Binding Functions and Forbidden Induced Subgraphs: A Survey

Bert Randerath and Ingo Schiermeyer
Graphs and Combinatorics 2019

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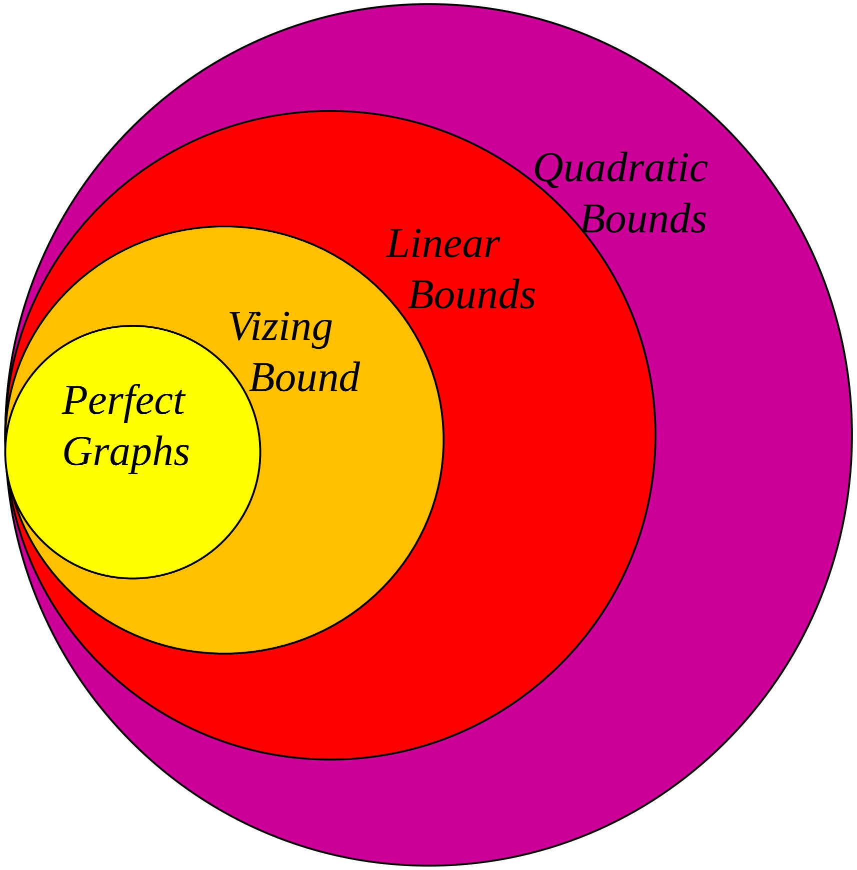
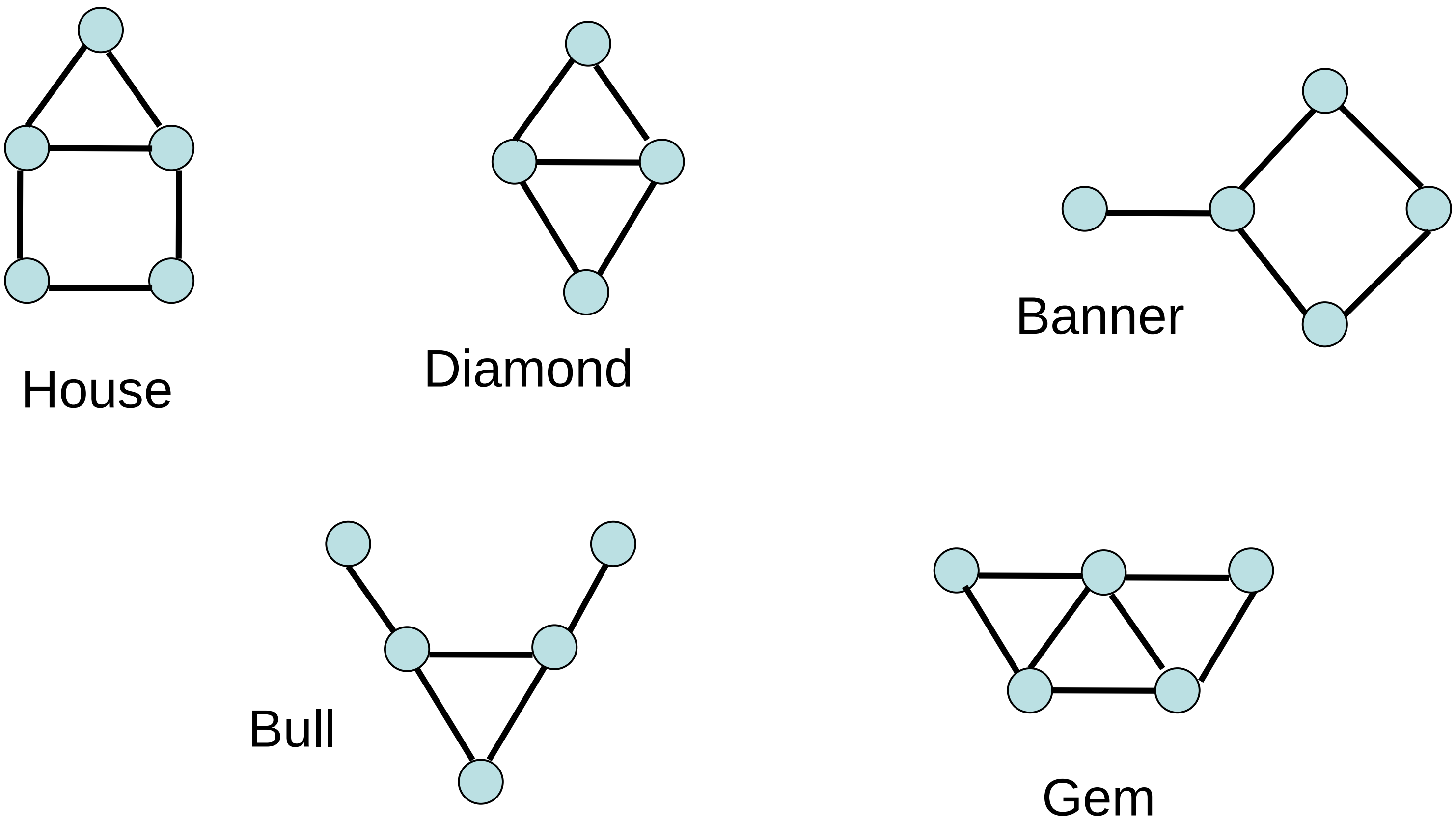


Induced subgraphs



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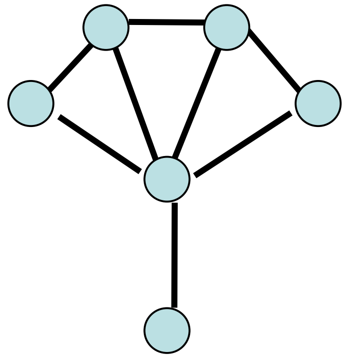
Theorem (Fouquet et al. 1995)

$$f(P_5, \text{House}) \le \binom{\omega}{2}$$



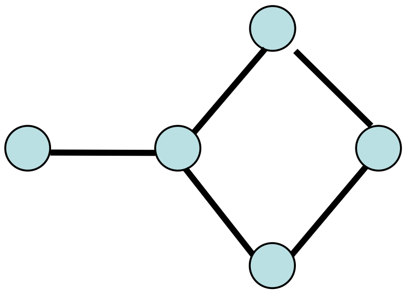
Parachute

Theorem (IS 2014)

$$f(P_5, H) \le \omega^2$$

$$\text{for } H \in \{\text{claw}, \text{paw}, \text{diamond}, \text{dart}, \text{gem}, \text{cricket}, \text{parachute}\}$$

Induced subgraphs



Banner

Theorem (Brause, Geisser, Randerath, and Schiermeyer 2019+)

$$c \bullet \frac{\omega(G)^2}{\log \omega} \leq f(P_5, \text{banner}) \leq \omega(G)^2$$

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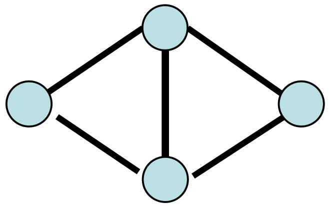
Theorem

$$f(P_5, \text{diamond}) \leq \omega(G) + 1$$



diamond

Theorem (Cameron, Huang, and Merkel 2018+)

$$f(P_6, \text{diamond}) \leq \omega(G) + 3$$

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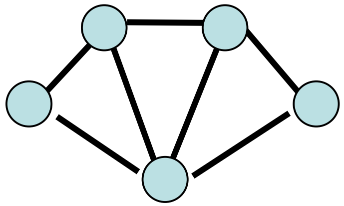
Theorem (Chudnovsky et al. 2019+)

$$f(P_5, \text{gem}) \leq \left\lceil \frac{5\omega(G)}{4} \right\rceil$$



gem

Theorem (Randerath and Schiermeyer 2019)

$$f(P_k, \text{gem}) \leq (k-2)(\omega(G)-1) \text{ for } k \geq 4$$

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Theorem (Chudnovsky and Sivaraman 2019)

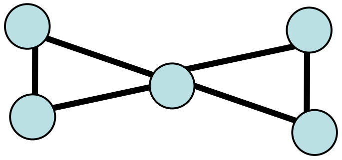
$$f(P_5, \text{bull}) \le \binom{\omega(G)+1}{2}$$

$$f(P_5, C_5) \le 2^{\omega(G)-1}$$

Problem: Can the chi-binding function for (P5,C5)-free graphs be improved?

Why is C5 so difficult?

Induced subgraphs



Butterfly -- Windmill (3,2)

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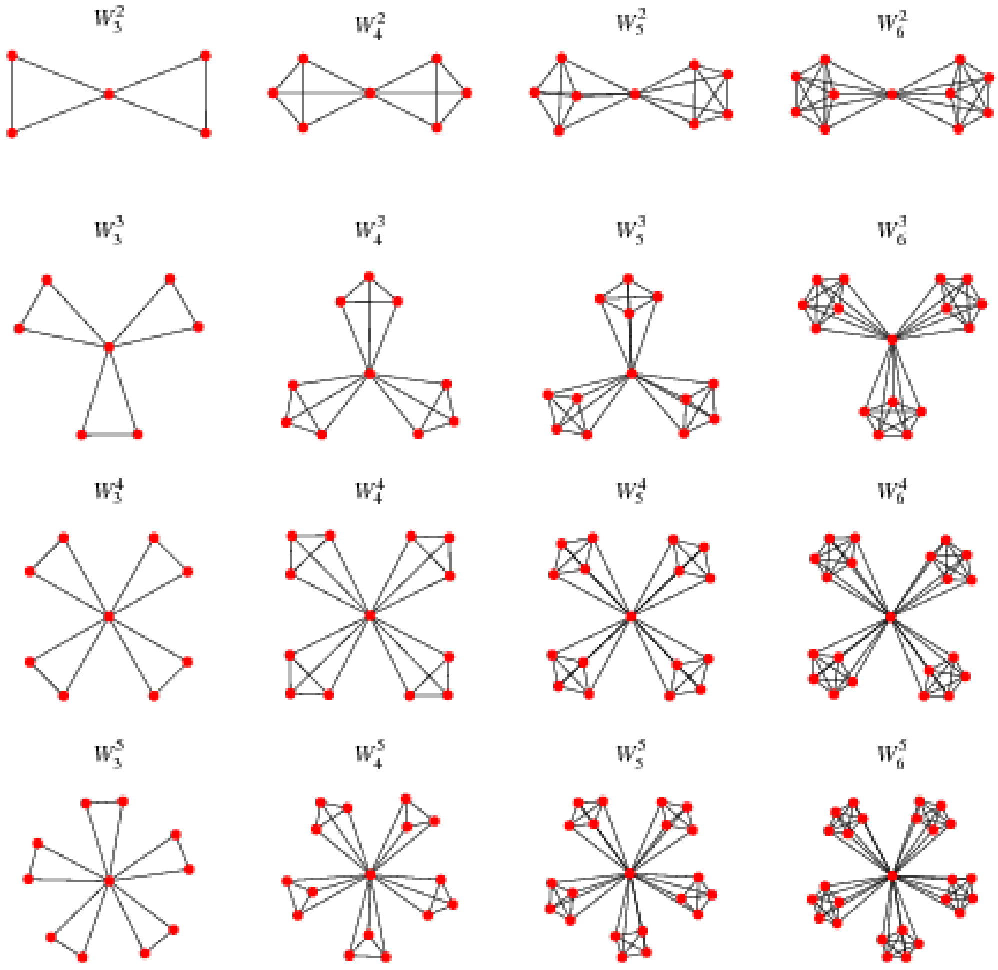
Windmill graphs



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Theorem (IS 2015)

Let H be the windmill W_3^p for some $p \geq 2$. Then $f(P_5, H) \leq \omega^{2p-1}$.

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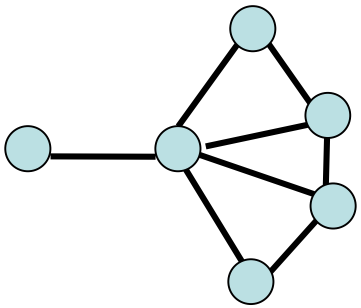
Theorem (Brause, Doan, and Schiermeyer 2015)

Let G be a $(P_5, K_{2,t})$-free graph for some $t \geq 2$.

Then $\chi(G) \leq c_t\, \omega^t$ for a constant c_t.

Induced subgraphs



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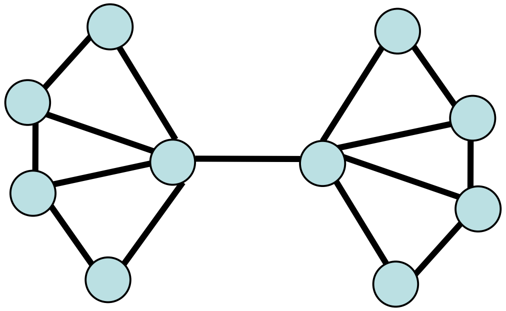

Twin-Parachute

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Problem

Does there exist a polynomial chi-binding function for (P_5, Twin-parachute)-free graphs?

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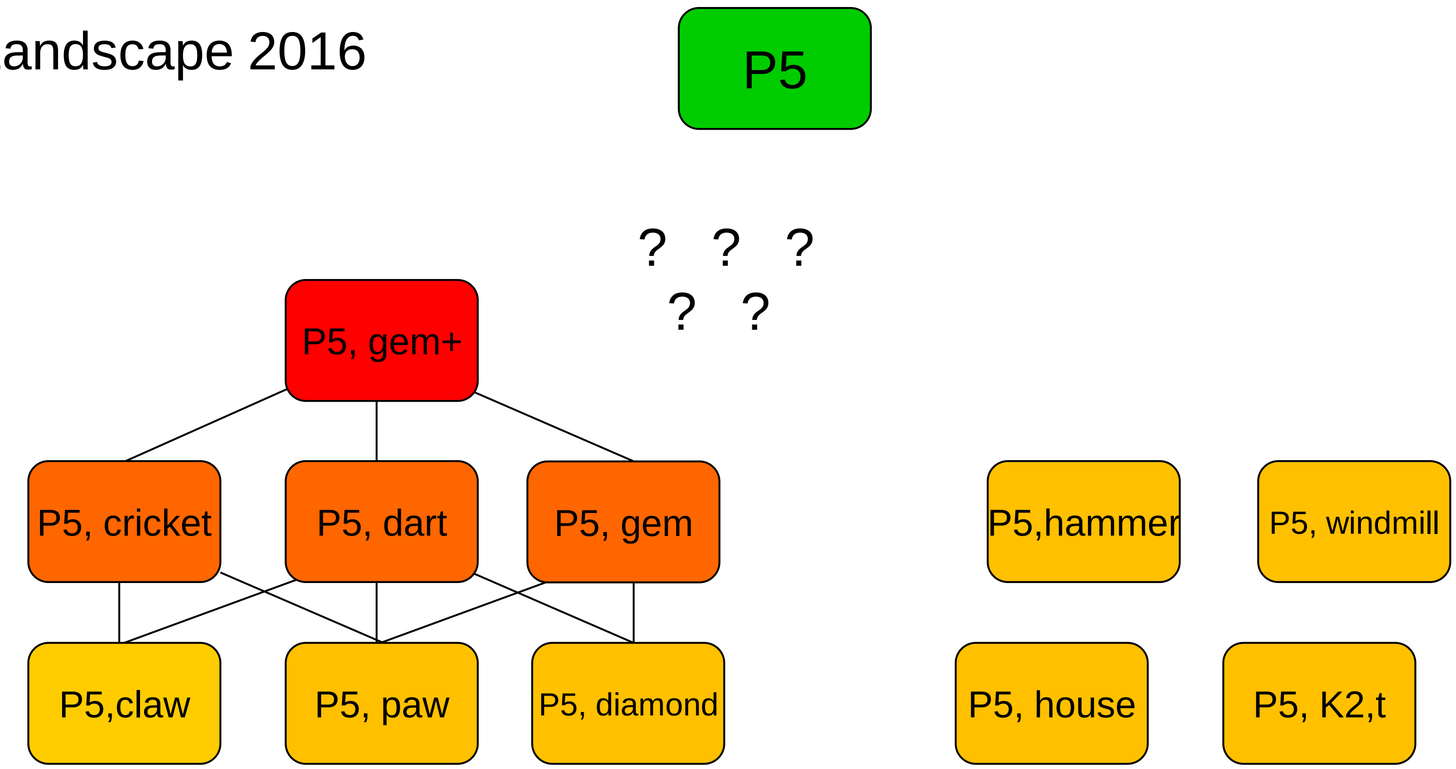
Landscape 2016



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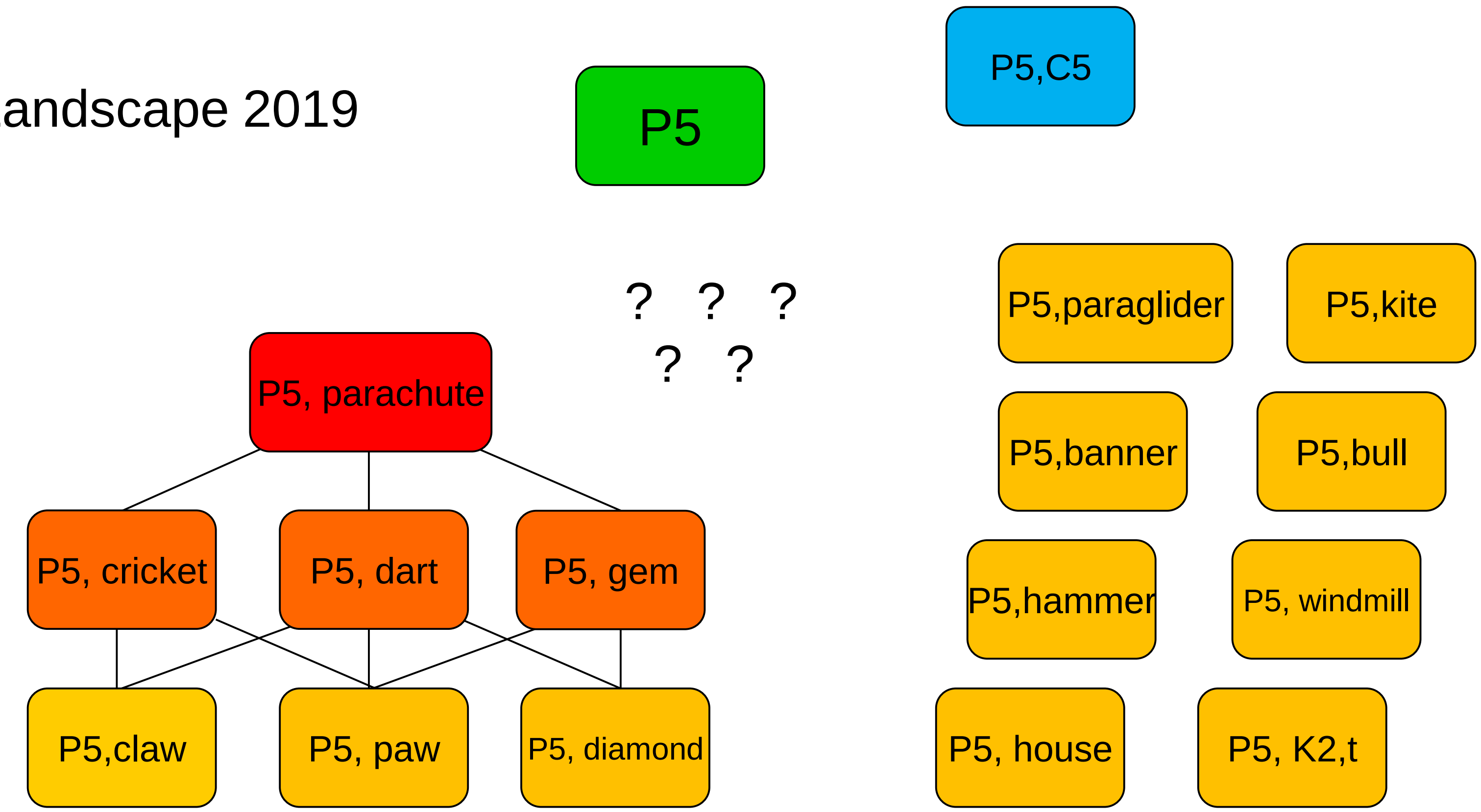
Landscape 2019



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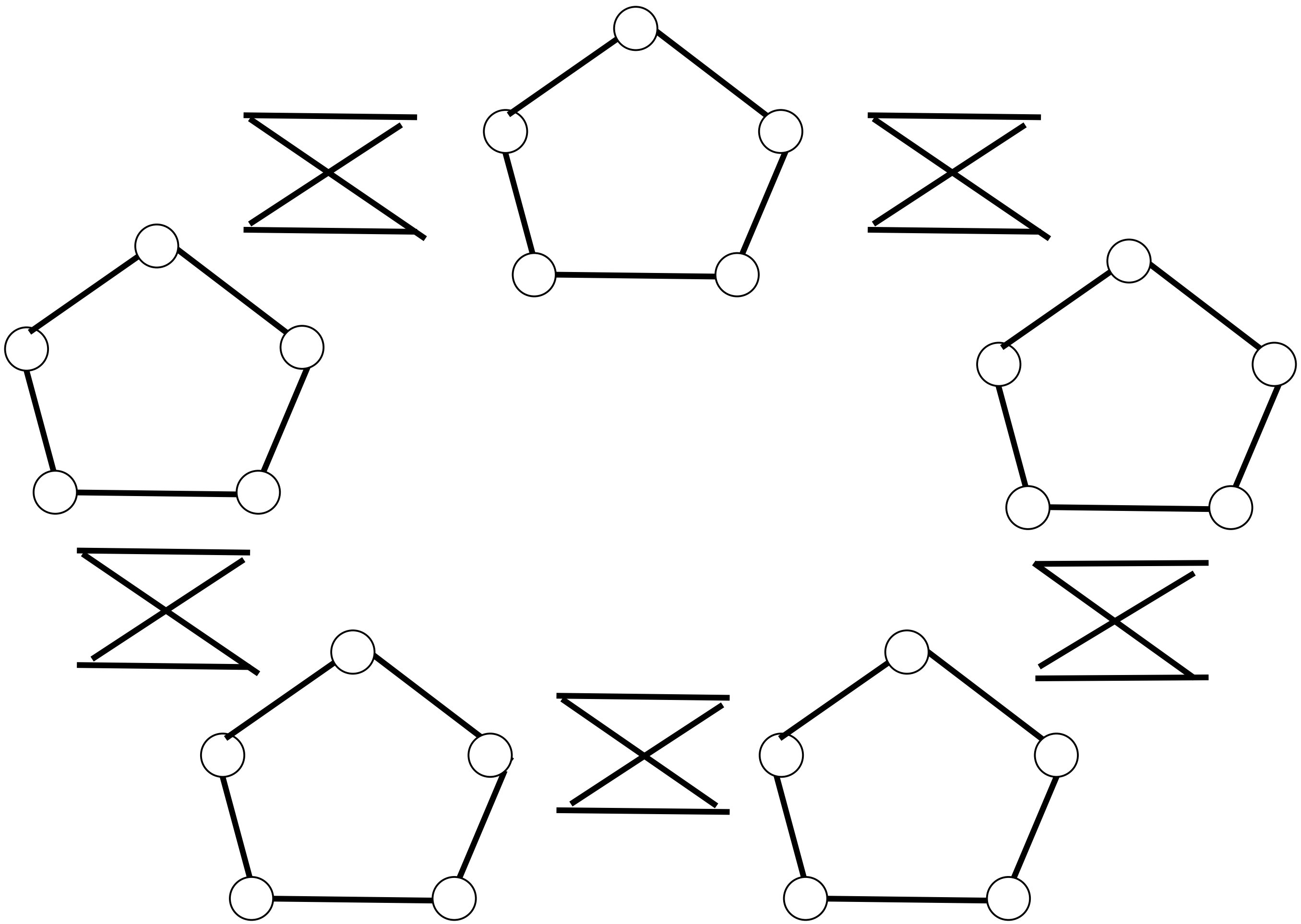

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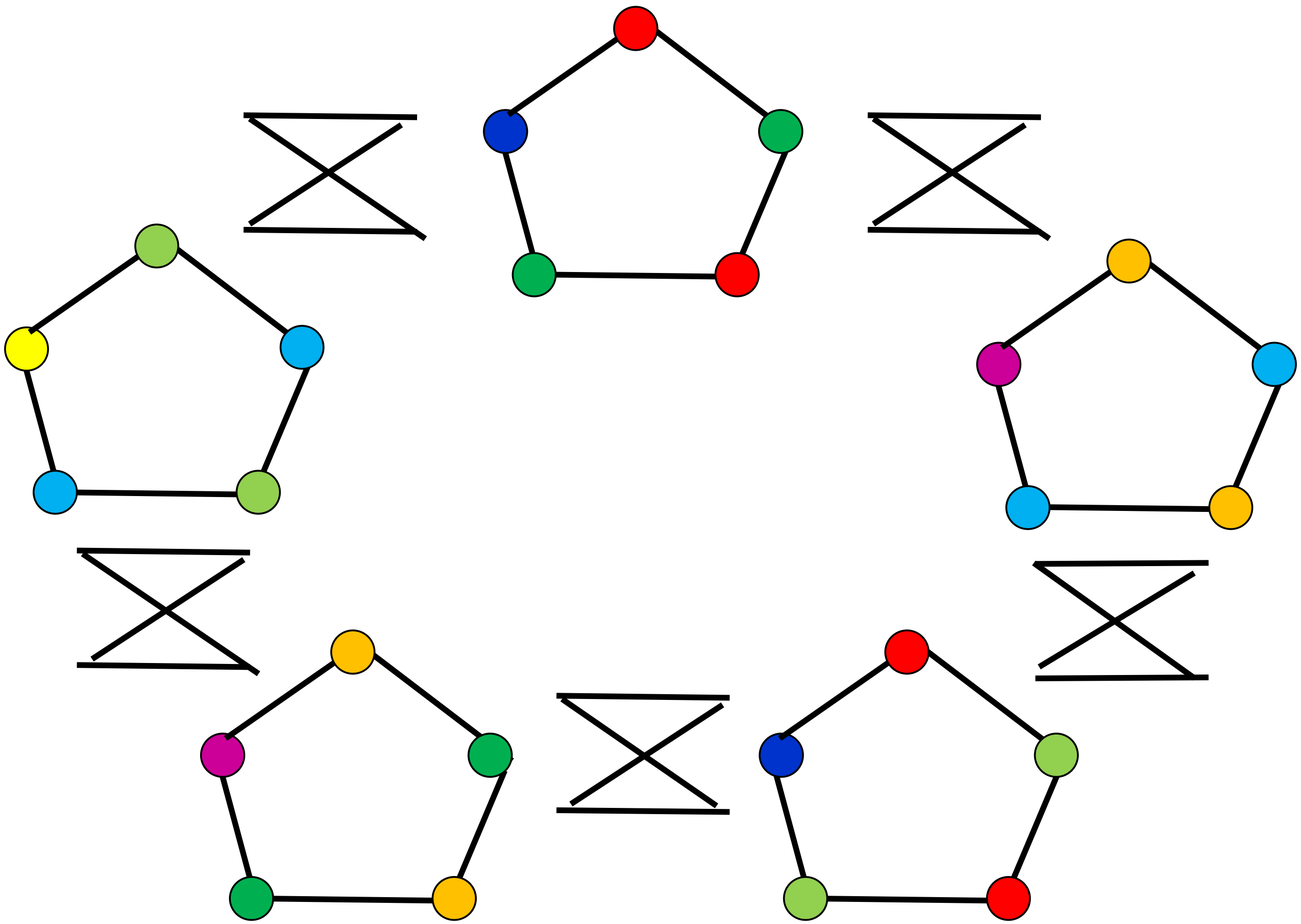

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Thank you very much!

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