The maximal abstract 3-rigidity matroid

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- It is **rigid** if every continuous motion of the vertices of (G, p) in $\mathbb{R}^d$, which preserves the lengths of its edges, also preserves the distances between all pairs of vertices.)

Rigidity: Example

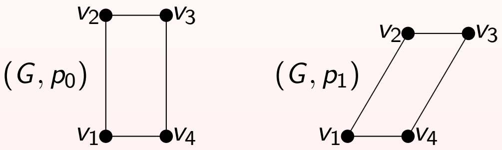

Figure: The 2-dimensional frameworks (G, p_0) and (G, p_1) are not rigid since (G, p_1) can be obtained from (G, p_0) by a continuous motion in $\mathbb{R}^2$ which preserves all edge lengths, but changes the distance between v_1 and v_3.

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- We say that a graph G is **rigid in** $\mathbb{R}^d$ if some (or equivalently every) generic realisation of G in $\mathbb{R}^d$ is rigid.
- The problem of characterising graphs which are rigid in $\mathbb{R}^d$ is solved for $d = 1$ (easy) and $d = 2$ (Pollaczek-Geiringer 1927, Laman 1970), but is open for $d \geq 3$.

The Rigidity Matrix

The rigidity of a given framework (G, p) is determined by the solution space of the system of quadratic equations

$$\|p_t(u) - p_t(v)\|^2 = d_{uv} \text{ for all } uv \in E \tag{1}$$

where $p_t(u)$ is the position of u at time t, $p_0 = p$, and $d_{uv} = \|p(u) - p(v)\|^2$.

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Differentiating (1) wrt t and putting $t = 0$, we obtain the following linear system of equations for the **instantaneous velocities** $\dot{p}(u)$ at time $t = 0$.

$$(p(u) - p(v)) \cdot (\dot{p}(u) - \dot{p}(v)) = 0 \text{ for all } uv \in E \qquad (2)$$

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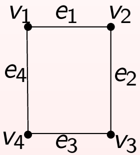
The **rigidity matrix** $R(G, p)$ of (G, p) is the matrix of coefficients of (2). It is an $|E| \times d|V|$ matrix with rows indexed by E and sequences of d consecutive columns indexed by V.

Example

The row of $R(G,p)$ indexed by $e = uv \in E$ is given by

$$e{=}uv \quad \left[\; 0\ldots0 \quad \overset{u}{p(u)-p(v)} \quad 0\ldots0 \quad \overset{v}{p(v)-p(u)} \quad 0\ldots0 \;\right].$$

Example



$$\begin{array}{c} \\ e_1 \\ e_2 \\ e_3 \\ e_4 \end{array} \begin{array}{c} \begin{array}{cccc} \quad v_1 \quad & \quad v_2 \quad & \quad v_3 \quad & \quad v_4 \quad \end{array} \\ \begin{pmatrix} p(v_1)-p(v_2) & p(v_2)-p(v_1) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & p(v_2)-p(v_3) & p(v_3)-p(v_2) & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & p(v_3)-p(v_4) & p(v_4)-p(v_3) \\ p(v_1)-p(v_4) & \mathbf{0} & \mathbf{0} & p(v_4)-p(v_1) \end{pmatrix} \end{array}$$

Infinitesimal Motions

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and (G,p) will be rigid if equality holds.
We say that (G,p) is **infinitesimally rigid** if

$$\text{rank } R(G,p) = \begin{cases} d|V| - \binom{d+1}{2} & \text{if } |V| \geq d+1 \\ \binom{|V|}{2} & \text{if } |V| \leq d+1 \end{cases}$$

Generic Rigidity and Independence

Theorem [Gluck, 1975]

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- The rigidity of (G, p) depends only on the graph G and the dimension d when (G, p) is generic.
- We can determine whether G is rigid in $\mathbb{R}^d$ if we can determine when a given set of rows of $R(G, p)$ is linearly independent when (G, p) is generic.

Matroids

A **matroid** $\mathcal{M}$ is a pair $(E, \mathcal{I})$ where E is a finite set and $\mathcal{I}$ is a family of subsets of E satisfying:

- $\emptyset \in \mathcal{I}$;
- if $A \subseteq B \subseteq E$ and $B \in \mathcal{I}$ then $A \in \mathcal{I}$;
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$A \subseteq E$ is **independent** if $A \in \mathcal{I}$ and A is **dependent** if $A \notin \mathcal{I}$. A is a **circuit** if it is a minimal dependent set. The **rank** of $\mathcal{M}$ is the size of a largest independent set in $\mathcal{M}$.

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We can define a partial order on the set of all matroids with the same groundset as follows. Given two matroids $\mathcal{M}_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$, we say $\mathcal{M}_1 \preceq M_2$ if $\mathcal{I}_1 \subseteq \mathcal{I}_2$.

The d-**dimensional rigidity matroid** $\mathcal{R}_d(G)$ of a graph $G = (V, E)$ is the matroid on E in which a set $F \subseteq E$ is **independent** if the rows of $R(G, p)$ indexed by F are linearly independent for some generic (G, p) in $\mathbb{R}^d$.

The d-dimensional rigidity matroid of G

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Note that if we can determine independence for all graphs in $\mathbb{R}^d$ then we can determine rigidity for all graphs in $\mathbb{R}^d$.

Abstract d-rigidity matroids: Jack Graver 1991

Let $\mathcal{M}$ be a matroid on $E(K_n)$ for some $n \geq d+2$. Then $\mathcal{M}$ is an **abstract d-rigidity matroid** if rank $M = dn - \binom{d+1}{2}$, and every $K_{d+2} \subseteq K_n$ is a circuit in $\mathcal{M}$ (Nguyen 2010).

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(a) There is a unique maximal abstract d-rigidity matroid on $E(K_n)$ for all d and all $n \geq d+2$.
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We show $\mathcal{C}_2^1(K_n)$ is the maximum abstract 3-rigidity matroid on $E(K_n)$ and characterise independence in this matroid.

The C_{d-1}^{d-2}-cofactor matroid

Let (G, p) be a framework in $\mathbb{R}^2$ and put $p(v_i) = (x_i, y_i)$ for $v_i \in V$. For $v_i, v_j \in E$ let $D_d(v_i, v_j) \in \mathbb{R}^d$ be defined by

$$D_d(v_i, v_j) = ((x_i - x_j)^{d-1}, (x_i - x_j)^{d-2}(y_i - y_j), \ldots, (y_i - y_j)^{d-1}).$$

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The C_{d-1}^{d-2}-**cofactor matrix** of (G,p) is the matrix $C_{d-1}^{d-2}(G,p)$ of size $|E|\times d|V|$ in which the row associated with the edge $e=v_iv_j$ with $i<j$ is

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$$\begin{array}{c} \\ e=v_iv_j \end{array} \begin{array}{c} \begin{matrix} & v_i & & v_j & \end{matrix} \\ \left[\begin{matrix} 0\ldots0 & D_d(v_i,v_j) & 0\ldots0 & -D_d(v_i,v_j) & 0\ldots0 \end{matrix} \right]. \end{array}$$

The C_{d-1}^{d-2}-**cofactor matroid** of G, $\mathcal{C}_{d-1}^{d-2}(G)$, is the row matroid of the cofactor matrix $C_{d-1}^{d-2}(G, p)$ for any generic p. We have:
$\mathcal{C}_{d-1}^{d-2}(K_n) = \mathcal{R}_d(K_n)$ for $d = 1, 2$.
$\mathcal{C}_{d-1}^{d-2}(K_n) \neq \mathcal{R}_d(K_n)$ for $d \geq 4$ and $n \geq 12$.

A K_5-**sequence in** K_n is a sequence of subgraphs $(K_5^1, K_5^2, \ldots, K_5^t)$ each of which is isomorphic to K_5.
It is **proper** if $K_5^i \not\subseteq \bigcup_{j=1}^{i-1} K_5^j$ for all $2 \leq i \leq t$.

The maximal abstract 3-rigidity matroid

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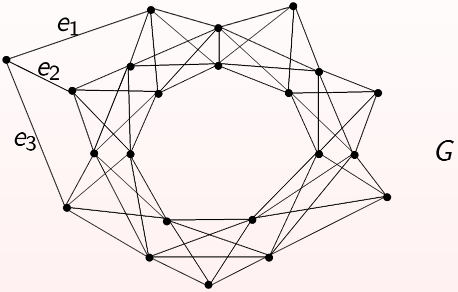
Theorem [Clinch, Tanigawa, BJ, 2019+]

(a) $\mathcal{C}_1^2$ is the unique maximal abstract 3-rigidity matroid on $E(K_n)$;
(b) $F \subseteq E(K_n)$ is independent in $\mathcal{C}_1^2$ if and only if

$$|F'| \leq \left| \bigcup_{i=1}^{t} E(K_5^i) \right| - t$$

for all $F' \subseteq F$ and all proper K_5-sequences $(K_5^1, K_5^2, \ldots, K_5^t)$ in K_n which cover F'.

Example



Let $F = E(G)$, $F' = F \setminus \{e_1, e_2.e_3\}$ and $(K_5^1, K_5^2, \ldots, K_5^7)$ be the 'obvious' proper K_5-sequence which covers F'. We have

$$57 = |F'| > \left| \bigcup_{i=1}^{7} E(K_5^i) \right| - 7 = 56$$

so F is not independent in $\mathcal{C}_2^1$.