Reduction of the Berge-Fulkerson Conjecture to cyclically 5-edge-connected snarks

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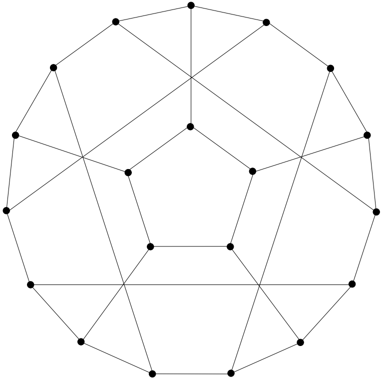
joint work with Edita Máčajová (Comenius University, Bratislava)

Berge-Fulkerson Conjecture

Conjecture (Berge-Fulkerson, 1971)

Every bridgeless cubic graph contains a family of SIX perfect matchings that together cover each edge exactly twice.



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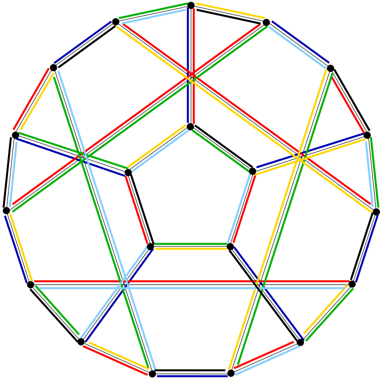

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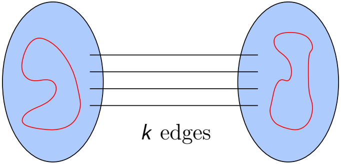
- Do we need to require a graph to be bridgeless?
  - YES! (a bridge in a cubic graph belongs to every perfect matching)
- ALTERNATIVE FORMULATION: if we double edges in a bridgeless cubic graph, we obtain a 6-edge-colourable 6-regular multigraph

Cyclic connectivity and oddness

Cyclic connectivity is the smallest number of edges which have to be removed in order to obtain at least two components containing cycles



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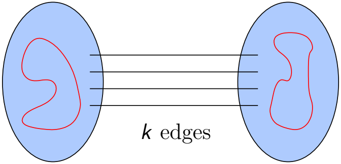

Conjecture (Jaeger, Swart'80)

There is no snark with cyclic connectivity greater than 6.

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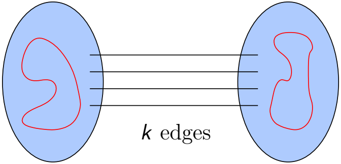

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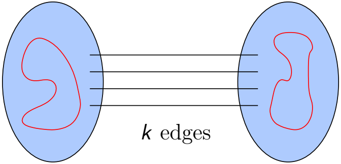

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- $\omega(G) = 0 \Leftrightarrow G$ is 3-edge-colourable

Possible Minimal Counterexamples to some Outstanding Conjectures

conj.	girth	cyclic connectivity	oddness
5–flow Conjecture	≥ 11 [Kochol]	≥ 6 [Kochol]	≥ 6 [GM, Steffen]
5–cycle double cover C.	≥ 12 [Huck]	≥ 4	≥ 6 [Huck]
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BF-colourings

Let G be a bridgeless cubic graph. Consider six perfect matchings of G, say $\{M_1, M_2, M_3, M_4, M_5, M_6\}$, such that every edge of G belongs to exactly two of them.

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These perfect matchings induce a map

$$\phi : E(G) \to \{ \textbf{ 2-subsets of } \{1, 2, 3, 4, 5, 6\}\}$$

$$\phi(e) = \{i, j\}, i \neq j$$

and

$$\phi(e) \cap \phi(f) = \emptyset$$

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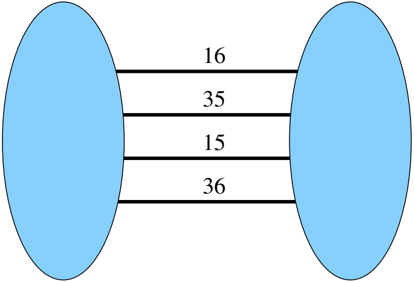
We say that ϕ is a *BF*-colouring of G.

BF-colourings of 4-poles



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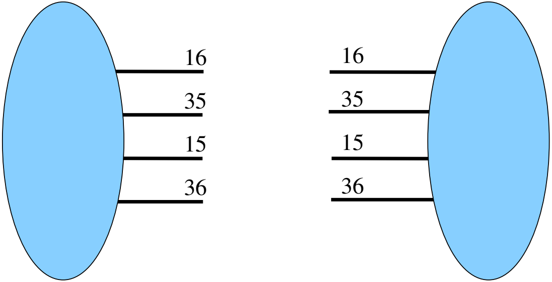

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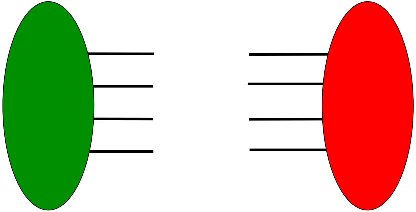

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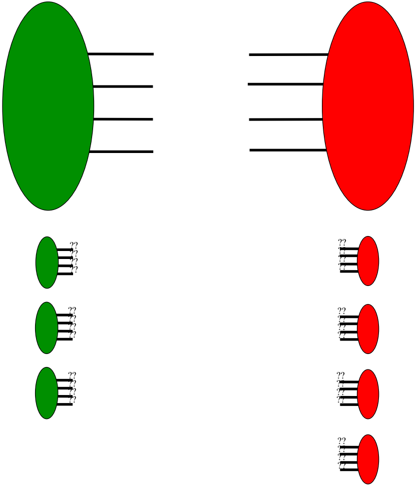

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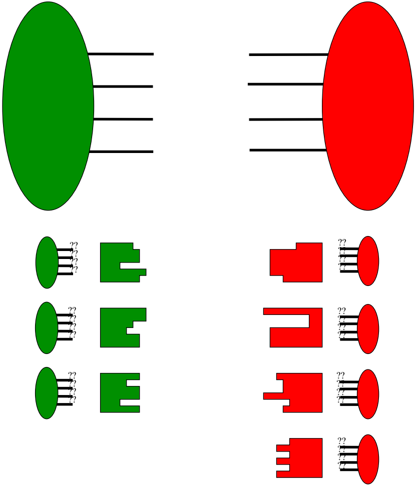

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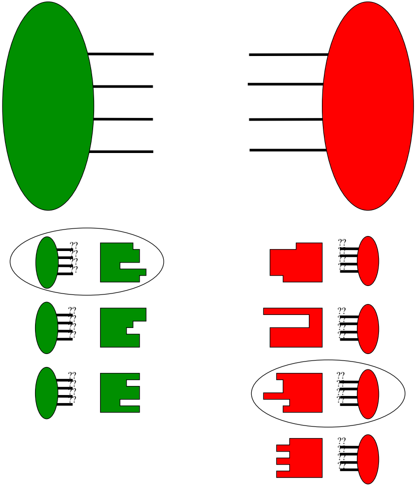

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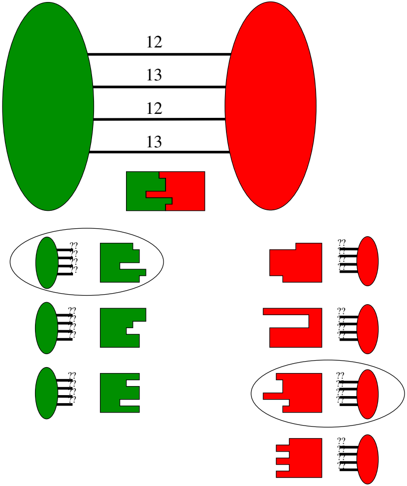
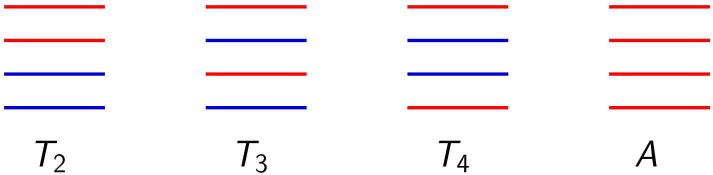


4-edge-cut colourings

There are exactly 4 types of possible partions of the 4 dangling edges along two disjoint perfect matchings:



"Splitting" of a BF-colouring of a 4-edge-cut

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1 2

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AA

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1 2	1 2
1 2	1 2
1 2	1 3
1 2	1 3
AA	AT_2

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1 2	1 2	1
1 2	1 2	1
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1 2	1 2	1
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- there exist $\binom{4}{2} + 4 = 10$ types of BF-colourings of a 4-edge-cut

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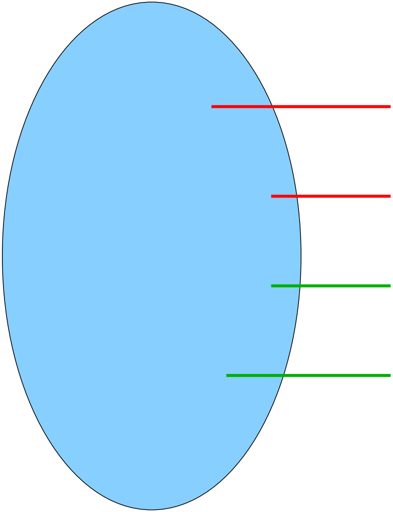

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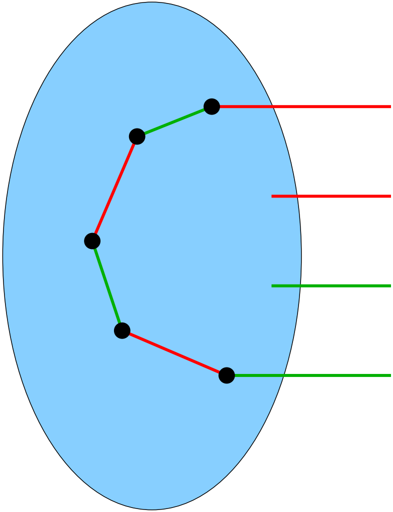

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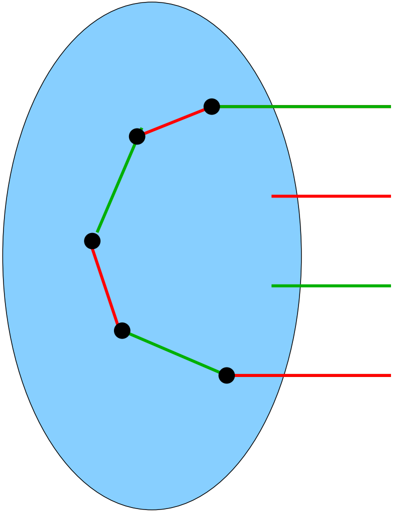

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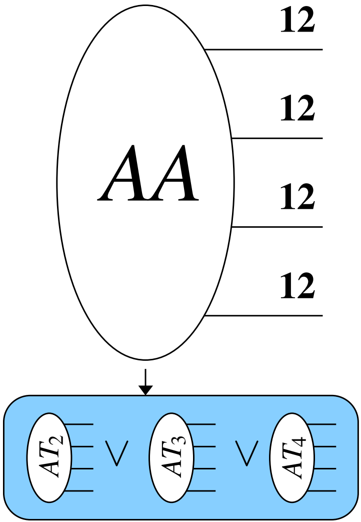

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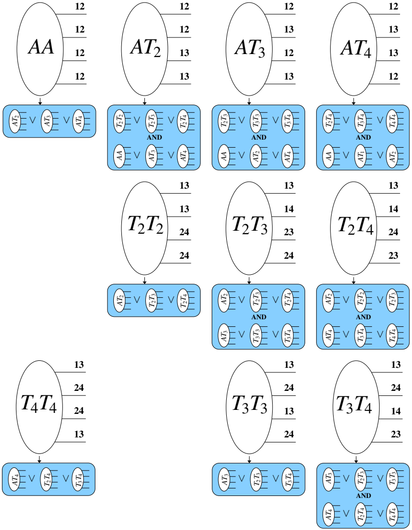
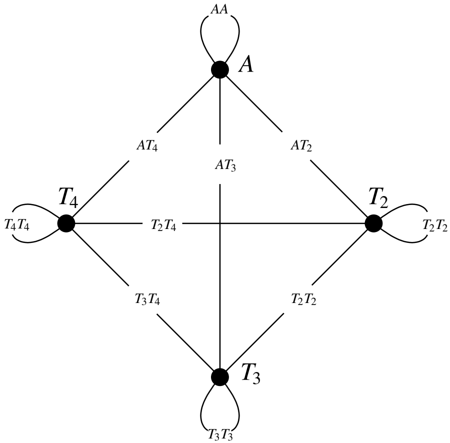
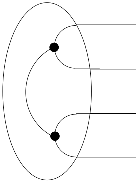


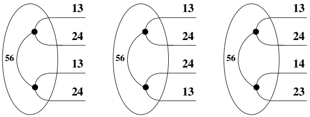
Graph of BF-colourings

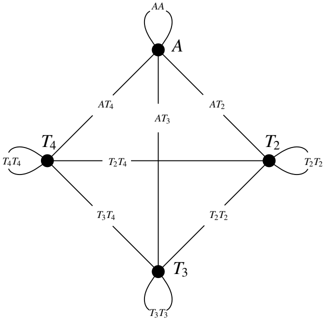
each 4-pole corresponds to a subgraph of M according to its admissible BF-colourings



4-pole $\rightarrow$ a subgraph of M







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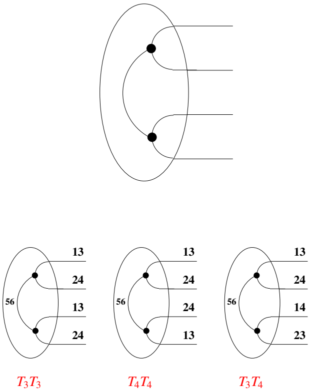
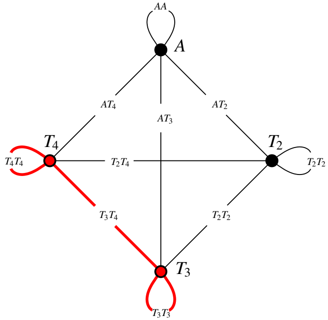
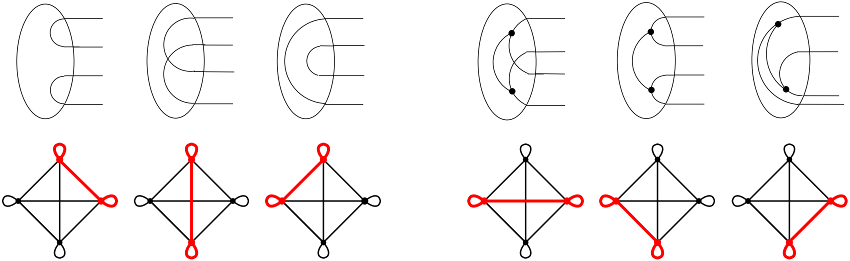




Acyclic 4-poles

There are only SIX different acyclic 4-poles. In each of them, the admissible BF-colourings correspond to one of the SIX dumbbell subgraphs of M.



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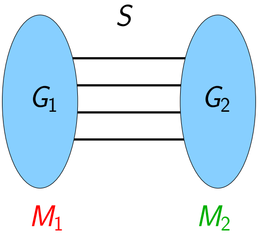
- assume that G is a smallest counterexample and that G contains a cycle separating 4-edge-cut S



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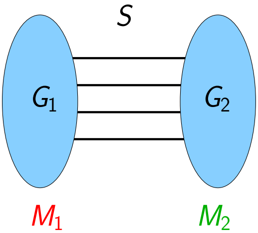

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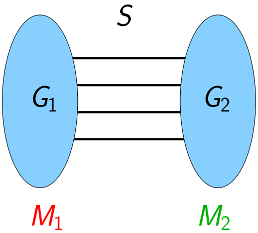

subgraphs of M

Sketch of the proof

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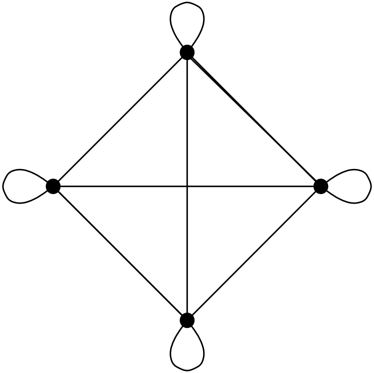

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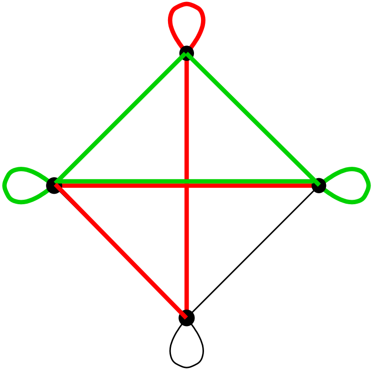

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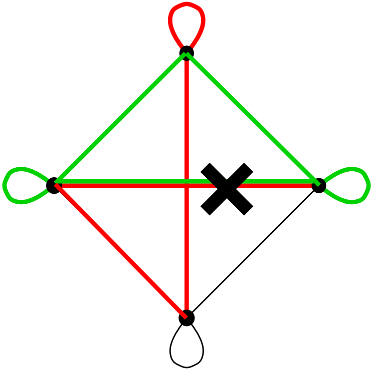

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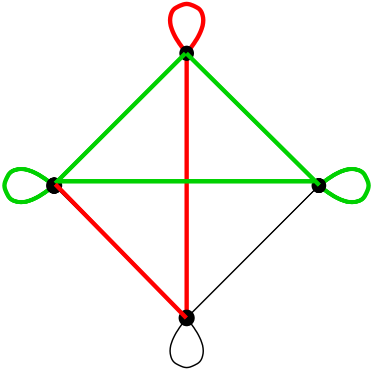

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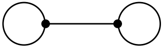
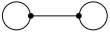
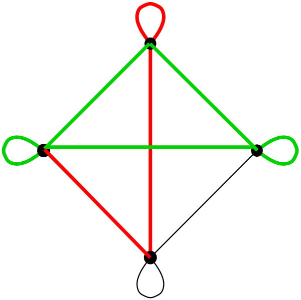
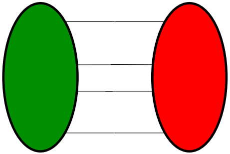
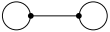
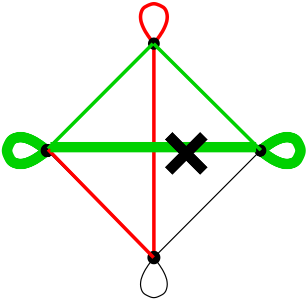
- neither M_i nor $\overline{M_i}$ contains a dumbbell subgraph 

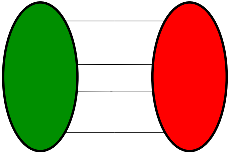
No  subgraph of M_i or $\overline{M_i}$

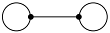


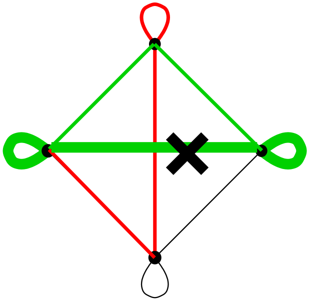


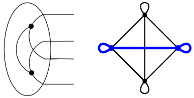
No  subgraph of M_i or $\overline{M_i}$

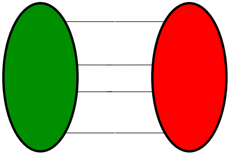


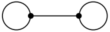


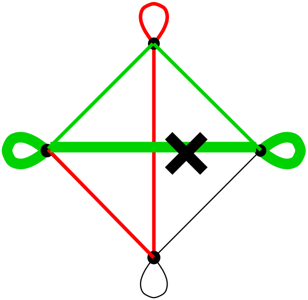
No  subgraph of M_i or $\overline{M_i}$

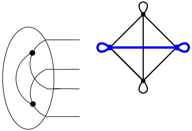


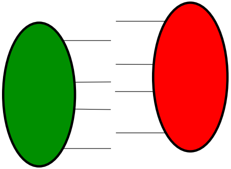


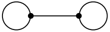


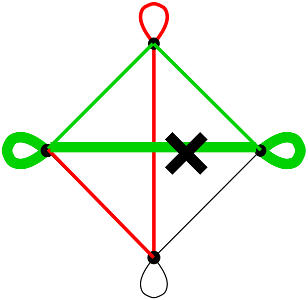
No  subgraph of M_i or $\overline{M_i}$

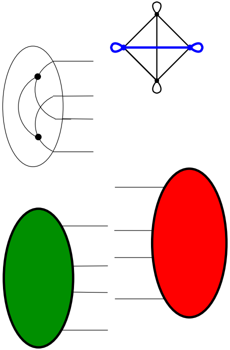


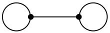


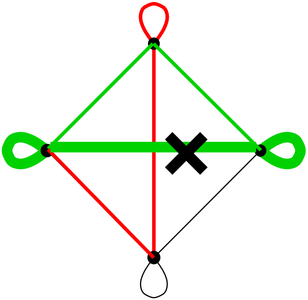


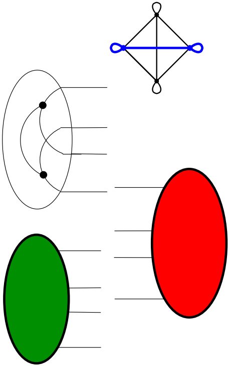
No  subgraph of M_i or $\overline{M_i}$

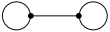


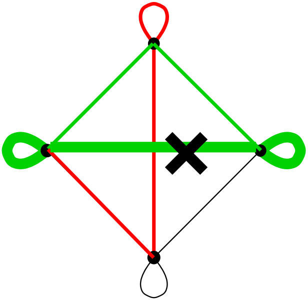


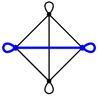
No  subgraph of M_i or $\overline{M_i}$

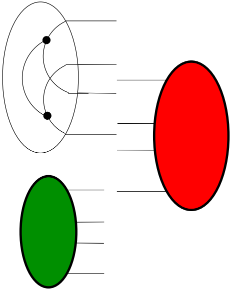


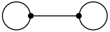


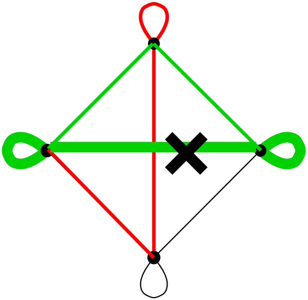
No  subgraph of M_i or $\overline{M_i}$

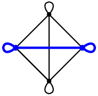


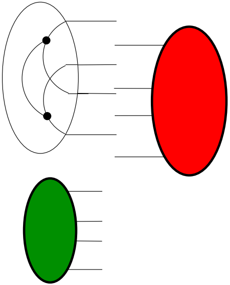


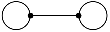


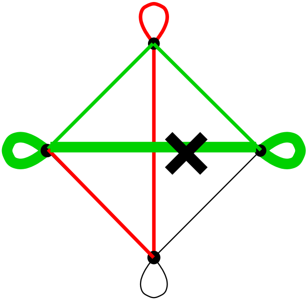
No  subgraph of M_i or $\overline{M_i}$

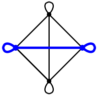


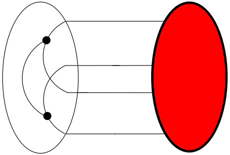


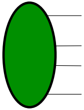


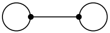
No  subgraph of M_i or $\overline{M_i}$

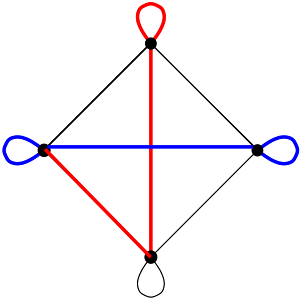


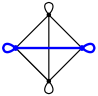




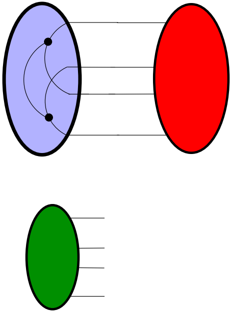


No  subgraph of M_i or $\overline{M_i}$





SMALLER COUNTEREXAMPLE!



Sketch of the proof

- M_1 and M_2 are edge-disjoint
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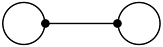
- neither M_i nor $\overline{M_i}$ contains a subgraph isomorphic to 

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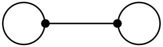
- neither M_i nor $\overline{M_i}$ contains a subgraph isomorphic to 
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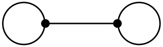
- neither M_i nor $\overline{M_i}$ contains a subgraph isomorphic to 
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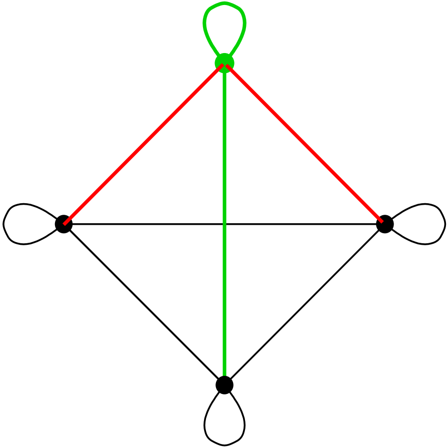
- further (and last) forbidden configuration....

a further forbidden configuration....



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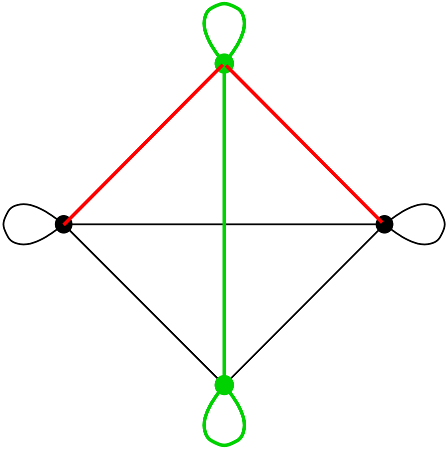

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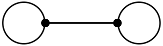
- neither M_i nor $\overline{M_i}$ contains a subgraph isomorphic to 
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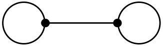
- neither M_i nor $\overline{M_i}$ contains a subgraph isomorphic to 
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...CONTRADICTION!

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https://combinatorics2020.unibs.it

List of plenary speakers

- Herivelto BORGES - University of San Paulo (Brasil)
- Bence CSAJBOK - Eotvos Lorand University (Hungary)
- Nicola DURANTE - University of Naples "Federico II" (Italy)
- Michel LAVRAUW - Sabanci University (Turkey)
- Patric R. J. OSTERGARD - Aalto University (Finland)
- Tomaz PISANSKI - Primorska University (Slovenia)
- Violet R. SYROTIUK - Arizona State University (USA)
- Ian WANLESS - Monash University (Australia)

Thank you for your attention!