Graphs with no short cycle covers

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joint work with Martin Škoviera

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Theorem (Bermond,Jackson,Jaeger 1983; Alon, Tarsi, 1985)

Every bridgeless graph G has a cycle cover of length at most $\frac{5}{3} \cdot |E(G)|$.

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- Chinese postman problem

$$scc(G) \geq cp(G)$$

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- [Brinkmann, Goedgebeur, Hägglund, Markström, 2013] up to 36 vertices there are two non-trivial cubic graphs that have the covering ratio greater than $\frac{4}{3}$ (the Petersen graph and G_{34} discovered by Hägglund in 2016)

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- both these graphs have $scc(G) = \frac{4}{3} \cdot |E(G)| + 1$
- [Esperet, Mazzuoccolo, 2014] infinite family with $scc(G) > \frac{4}{3} \cdot |E(G)|$
- [Esperet, Mazzuoccolo, 2014] there exists G with $scc(G) \geq \frac{4}{3} \cdot |E(G)| + 2$

SCCC and cubic graphs

Conjecture (Brinkmann, Goedgebeur, Hägglund, Markström, 2013)

For every cyclically 4-edge-connected cubic graph G with m edges

$$scc(G) \leq \frac{4}{3} \cdot m + o(m).$$

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we disprove the conjecture:

Theorem (EM,Škoviera)

There exists a family of cyclically 4-edge-connected cubic graphs G_n, $n \geq 1$ such that

$$scc(G_n) \geq (\frac{4}{3} + \frac{1}{69})|E(G_n)|.$$

SCC and pmi

G – bridgeless cubic graph

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- pmi(G) $=$ the minimum number of perfect matchings that cover all the edges of G
- [Hägglund, Markström, 2013; Steffen 2015] if $scc(G) > \frac{4}{3}$ then $pmi(G) = 5$

Sketch of the proof

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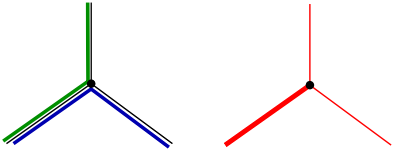
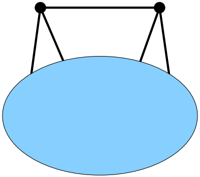
for each cubic graph G, $scc(G) \geq \frac{4}{3} \cdot |E(G)|$

- a cubic graph G such that $scc(G) = \frac{4}{3} \cdot |E(G)|$ is called light



Resistant $(2, 2)$-pole



no light cover

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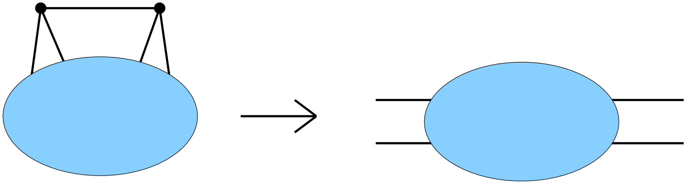

no light cover $\longrightarrow$ resistant $(2, 2)$-pole

Properties of a resistant $(2, 2)$-pole

$S(e)$ – set of cycles containing e

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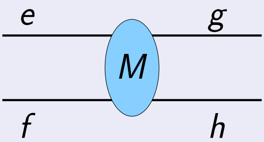
Lemma

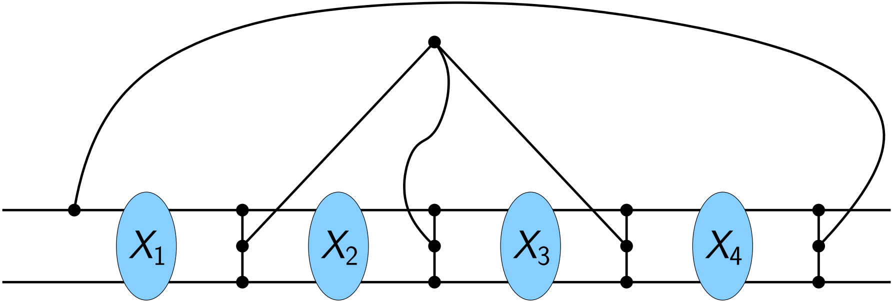
M = resistant $(2, 2)$-pole



In every light cover

- *if $w(M) = 4$, then $S(e) = S(f)$ and $S(g) = S(h)$*
- *if M is of type C then $S(e) \cap S(f) = \emptyset$ and $S(g) \cap S(h) = \emptyset$*
- *if $w(e) = w(f) = 1$ and e and f belong to the different cycles of the cycle cover, then $w(g) = w(h) = 2$*

$(2,2)$-pole Z



X_1, X_2, X_3, X_4 – resistant $(2,2)$-poles

Z is not a light $(2, 2)$-pole

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- for $i \in \{2,3\}$, the weight of each X_i is 4 or 6

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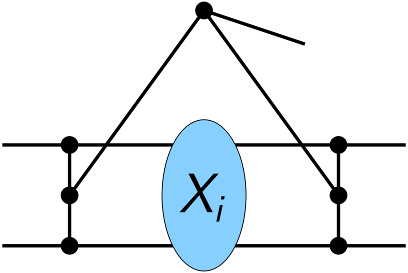

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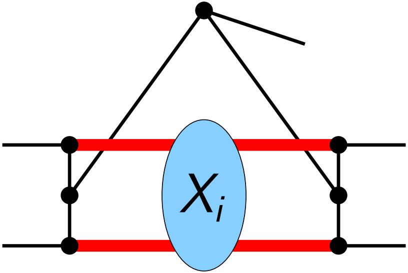

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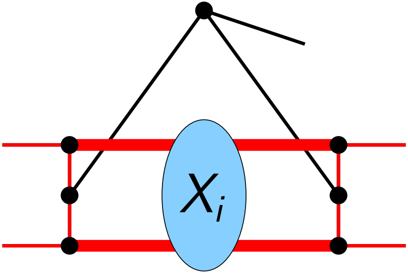

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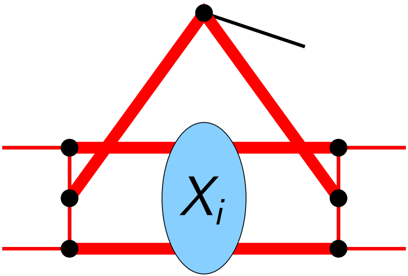

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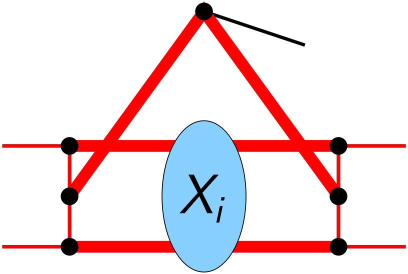
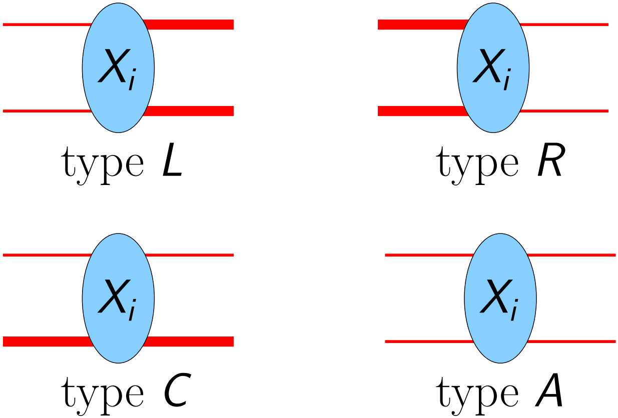


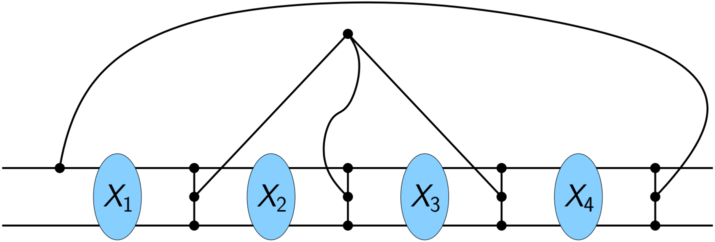
a contradiction

Types of X_2 and X_3

- each of X_2 and X_3 is of one of the types A, L, R, C



Classification by $(\alpha, \beta) = (\textit{type of } X_2, \textit{ type of } X_3)$



- $\alpha, \beta \in \{A, C, L, R\}$ (16 possibilities)

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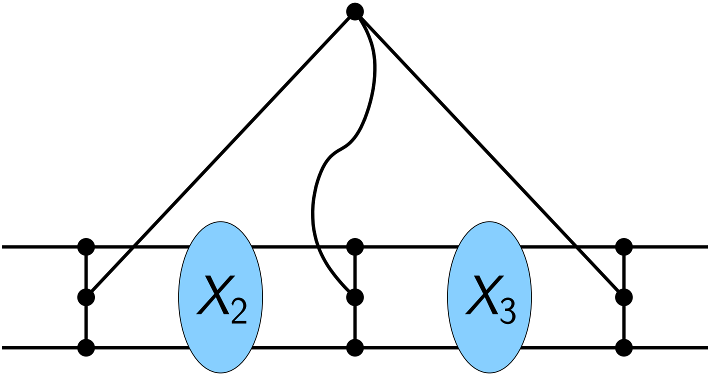

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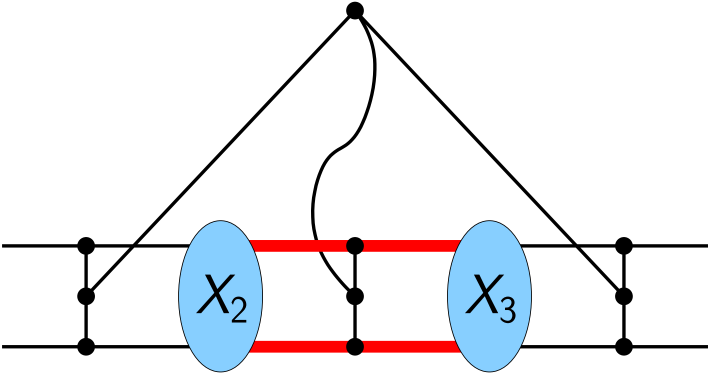

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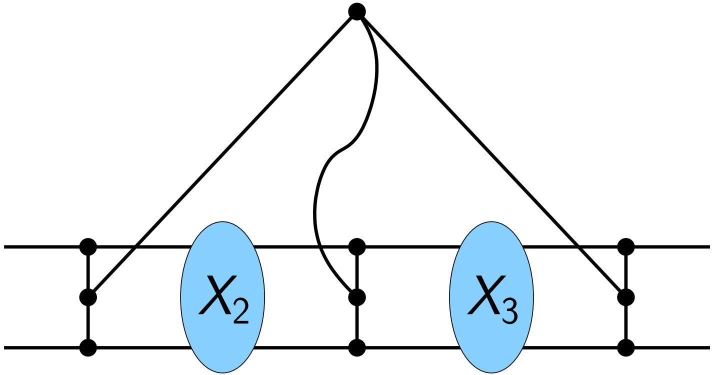

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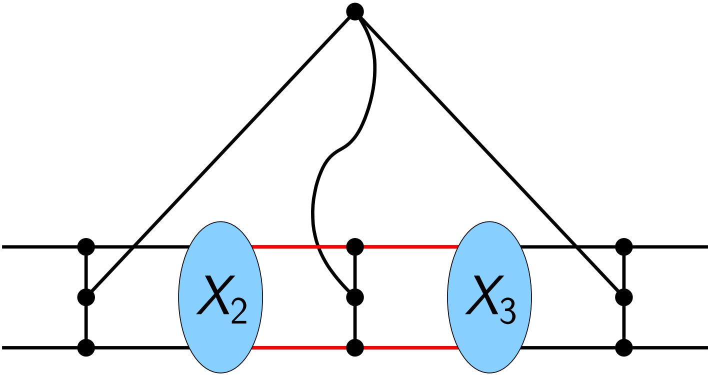

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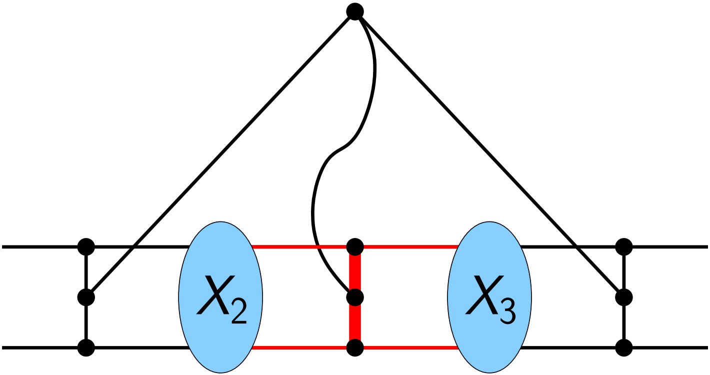

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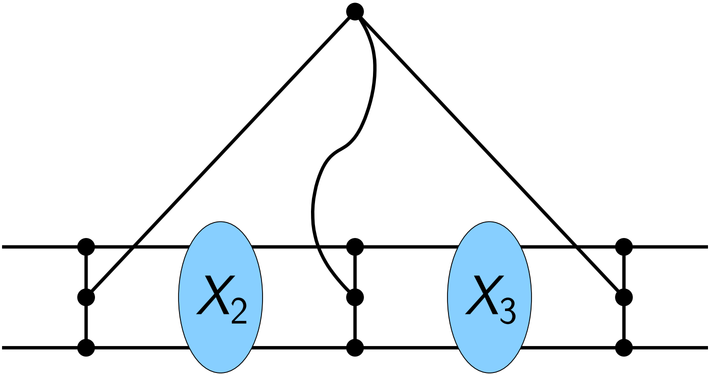

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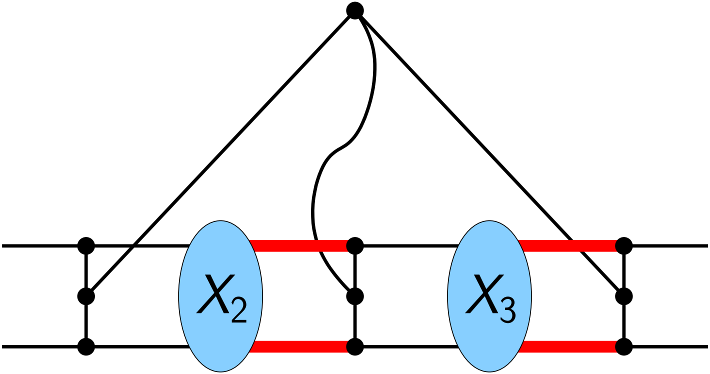

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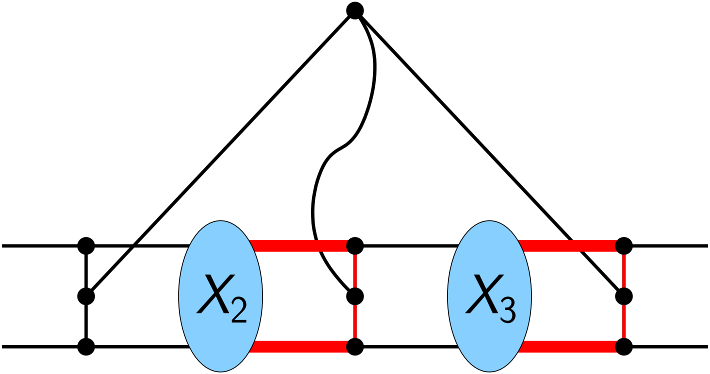

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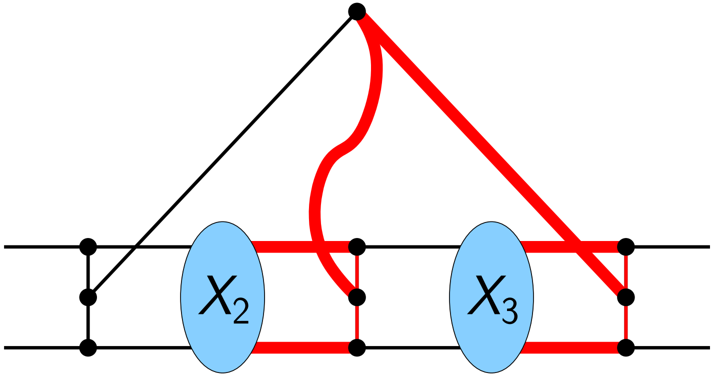

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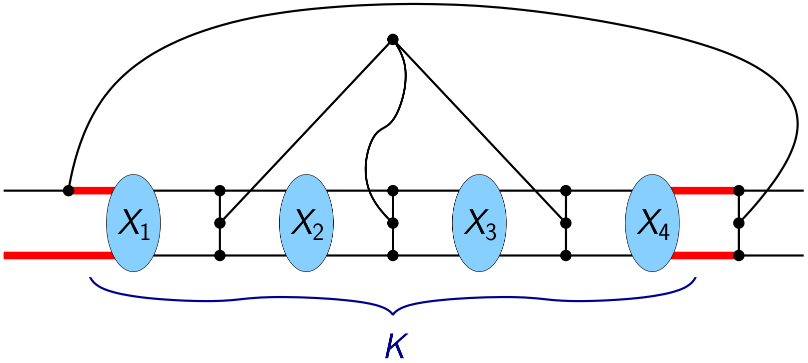

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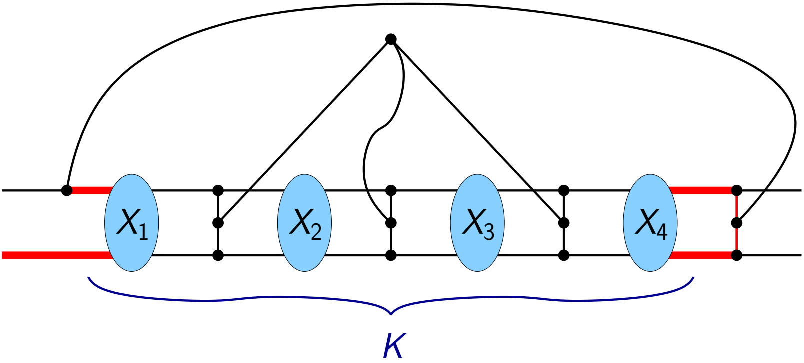

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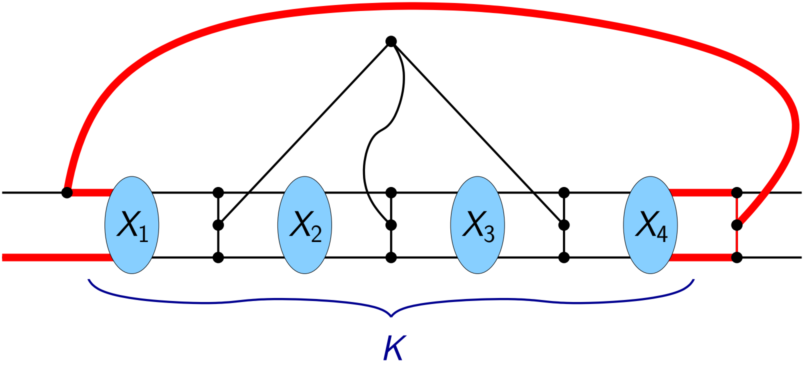

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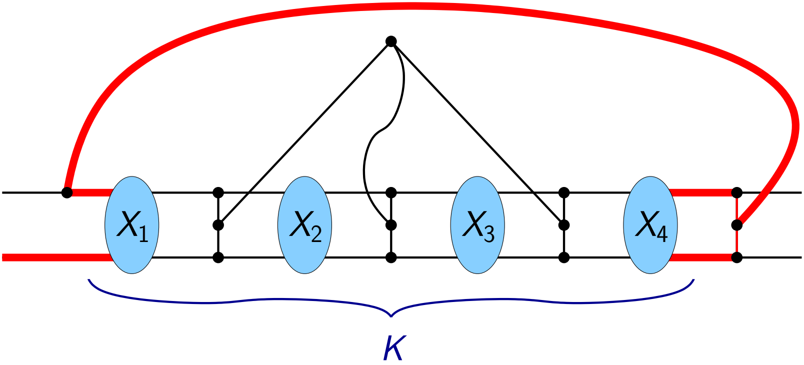

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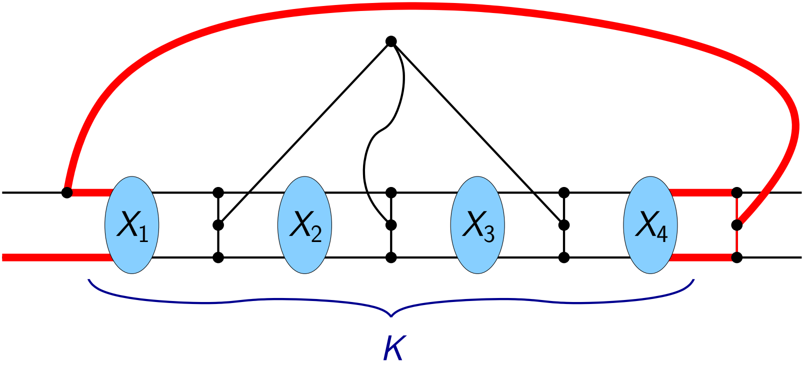

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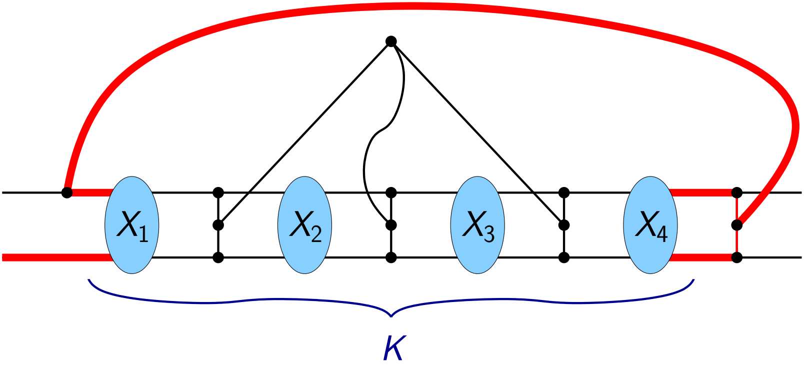

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- $scc(G_k) \geq (\frac{4}{3} + \frac{1}{69})|E(G_k)|$

Conclusion

Conjecture (Brinkmann, Goedgebeur, Hägglund, Markström, 2013)

For every cyclically 4-edge-connected cubic graph G with m edges,

$$scc(G) \leq \frac{4}{3} \cdot m + o(m).$$

Conjecture (EM, Škoviera)

For every cyclically 5-edge-connected cubic graph G with m edges,

$$scc(G) \leq \frac{4}{3} \cdot m + o(m).$$

Thank you for your attention!