Toward a structure theory of crossing-critical graphs

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on Structure and Algorithms
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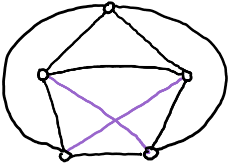
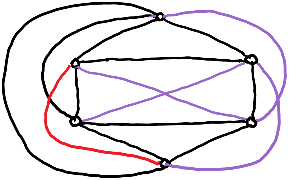
Crossing number of a graph

Crossing number of G ... $\boxed{cr(G)}$

Minimum number of crossings of edges when G is drawn in the plane



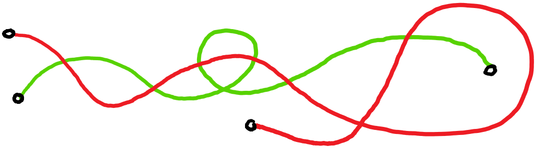
$cr(K_5) = 1$,



$cr(K_6) \leq 3$

Do we understand it well?

Is the crossing number equal to the pair-crossing number?



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Known to be true for $n \leq 6$ and for $K_{7,m}$ ($7 \leq m \leq 10$), Woodall (1993), computer-assisted proof. Open for $K_{9,9}$ and $K_{7,11}$?

Crossing-Critical Graphs

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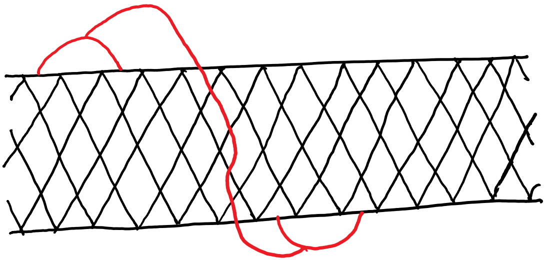
Conjecture (Richter)

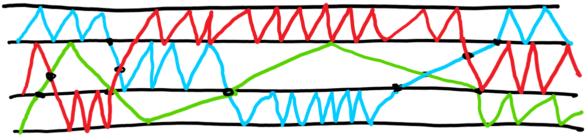
If G is c-CC, then $cr(G) \leq c + \Theta(\sqrt{c}\,)$.

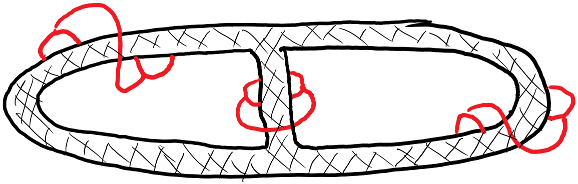
Constructions for a fixed c

- Širan (1984)
- Salazar (2003)
- Hlineny (2002)



More general examples





Toward a global structure

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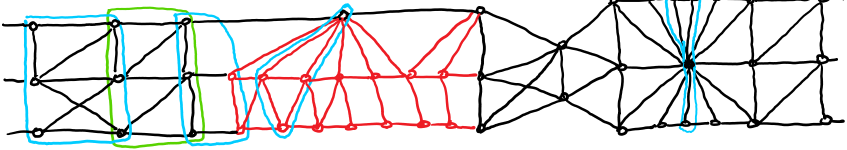
Theorem (Hlineny 2003)

For every fixed c, the c-CC graphs have bounded path-width, $pw(G) \leq 2^{2000c^3 \log c}$.



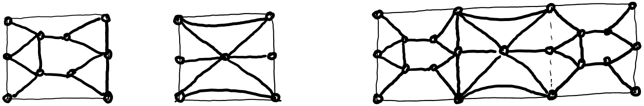
It was conjectured that c-CC graphs also have bounded bandwidth (as suggested by known examples).

Big Degrees

Dvorak & Mohar (2010): There are c-CC graphs with vertices of arbitrarily large degrees.

Recent result (SoCG 2019): Large degrees in c-CC graphs are possible for every $c \geq 13$ but do not occur when $c \leq 12$.

Tiles

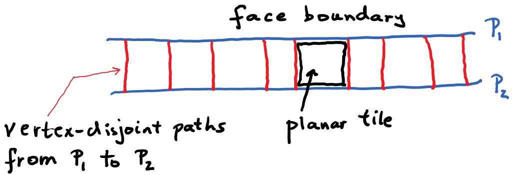


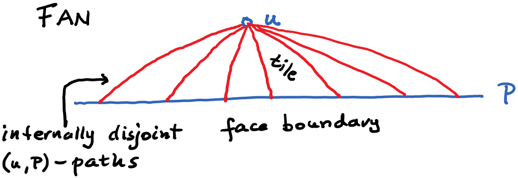
Theorem (Bokal, Oporowski, Richter, Salazar 2016)

There is a constant k such that every 2-CC graph contains a set of at most k vertices whose removal leaves a graph, each of whose components is a long sequence of 42 types of tiles.

Local structure – Bands and Fans

Consider an optimal drawing of a c-CC graph.





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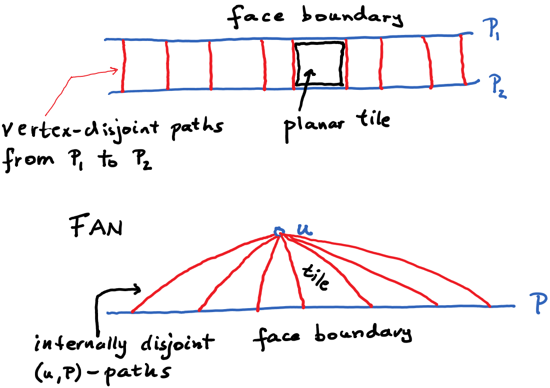


Theorem

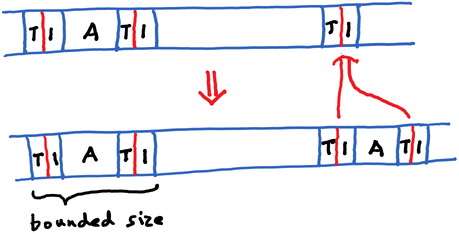
Any optimal drawing of a large c-CC graph contains a crossing-free subdrawing that is isomorphic to a large band or a large fan.

Main Theorem – Structure and generation

Theorem (Dvorak, Hlineny, M., SoCG 2018)

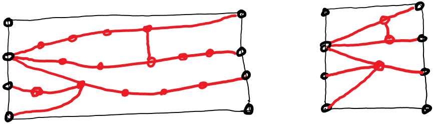
(a) For every $c \geq 2$ there is a finite set of c-CC graphs $F_1, \ldots, F_{N(c)}$ and every other c-CC graph can be obtained from one of these by replicating tiles of bounded size.

(b) Moreover, all graphs obtained through the replication process are c-CC.



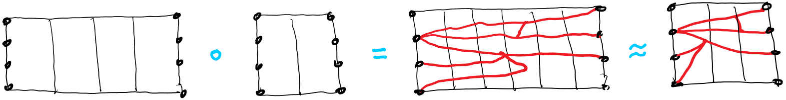
About the proof: Tiles as a semigroup

Two tiles are q-equivalent if they contain precisely the same linkages of type q.



Theorem

Composition of tiles forms a finite semigroup with respect to the q-equivalence.



Simon's factorization forest

A a finite semigroup

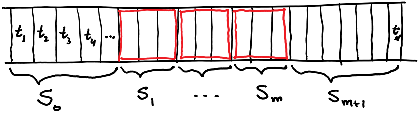
s a long product, want to express it as a long product in which many factors are repeated.

Theorem

$\forall A : \forall f : \mathbb{N} \to \mathbb{N} : \exists k_0, n_0$ such that every product $\boxed{s = t_1 t_2 \cdots t_N}$ *with $N \geq n_0$, the sequence can be partitioned into substrings,* $\boxed{s = S_0 S_1 S_2 \ldots S_m S_{m+1}}$ *such that*

(a) $m \geq f(k)$ and each S_i $(1 \leq i \leq m)$ has length at most k, and

(b) the product of elements in each S_i $(1 \leq i \leq m)$ is the same idempotent element of A.

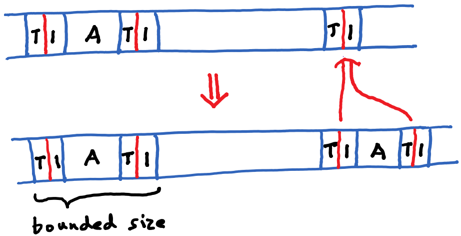


$S_1 \approx S_2 \approx \cdots \approx S_m$

$S_i^2 = S_i \,, \quad i = 1, 2, \ldots, m$

About the proof (simplified explanation)

- Suppose G is very large, consider one of its optimal drawings.
- Find a long band or fan.
- Show there is a lot of repetition of certain short parts.
- Find a repeated part that can be reduced (inverse of replication).
- Show that the resulting graph is c-CC if and only if G is c-CC. (For this proof to work we need a slightly more complicated replication condition.)



Questions?