Circumference of essentially 4-connected planar graphs

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joint work with

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Introduction

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essential connectivity

- A 3-connected planar graph G is *essentially 4-connected* if every 3-separator of G is trivial.

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3-connected planar graphs

For every 3-connected planar graph G,

- $\text{circ}(G) \geq cn^{\log_3 2}$, for some $c \geq 1$ [Chen, Yu, 2002]

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essentially 4-connected planar triangulations

For every essentially 4-connected planar triangulation G,

- $\mathrm{circ}(G) \geq \frac{13}{21}(n+4)$ [F., Harant, Jendroľ, 2016]

Sharpness of a lower bound on circ

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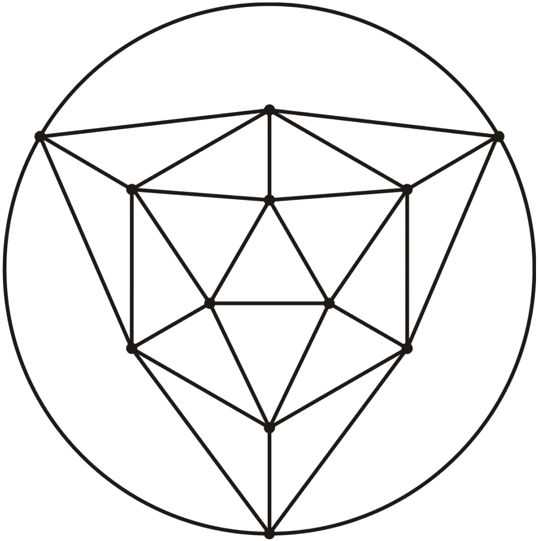
Construction



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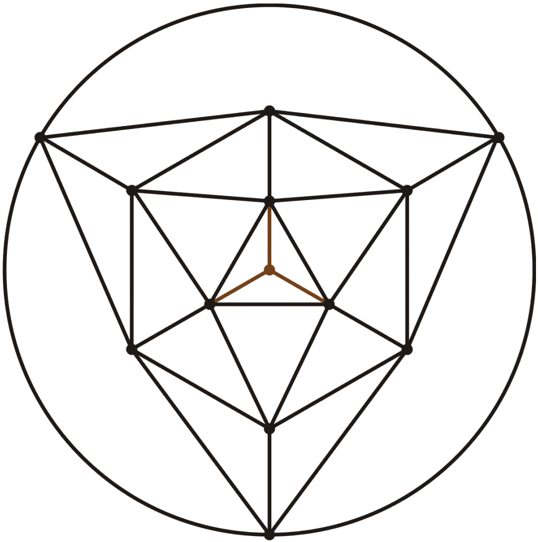

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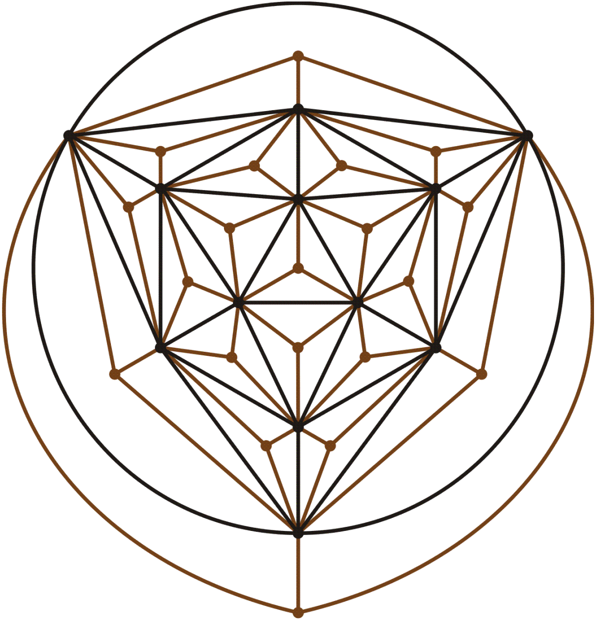

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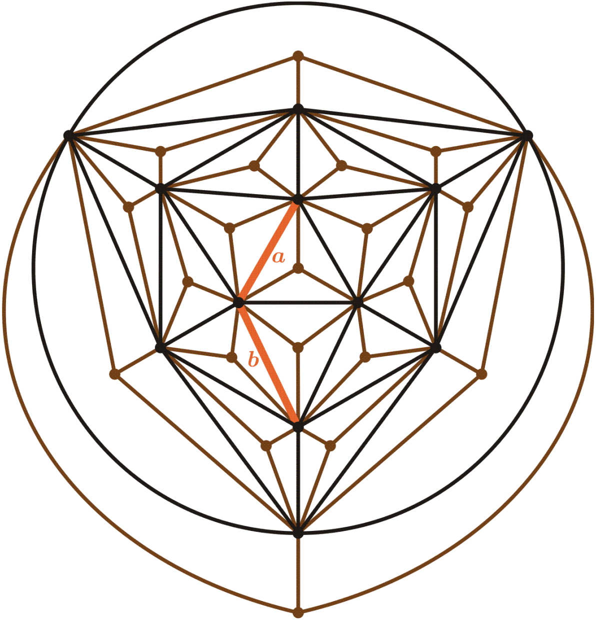

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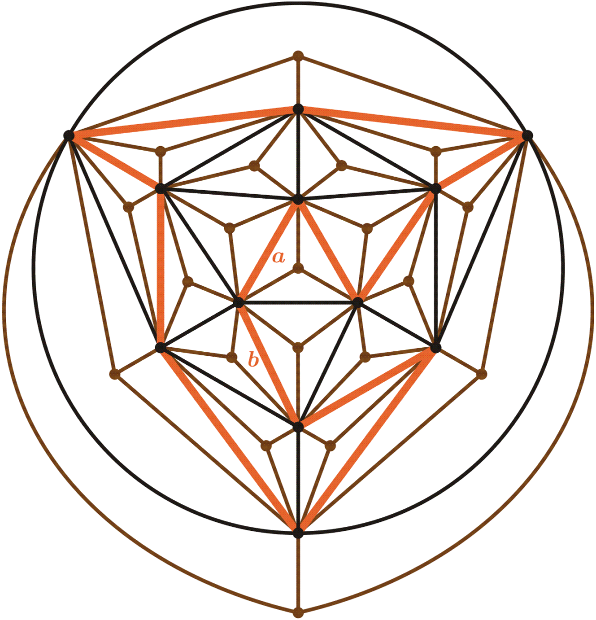

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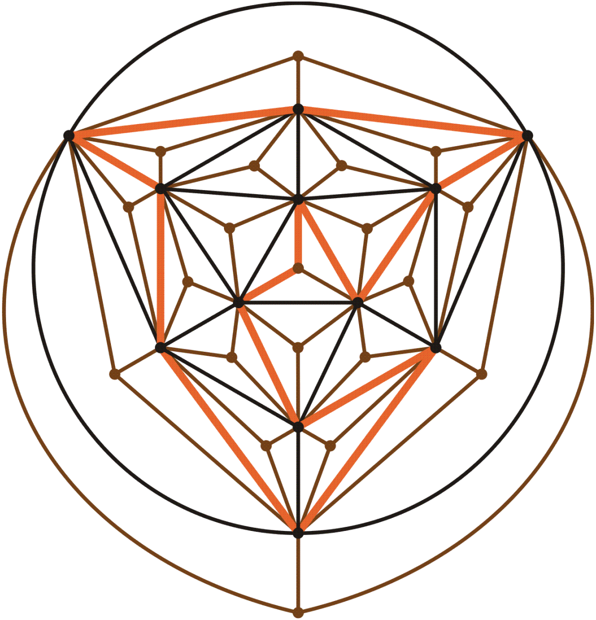

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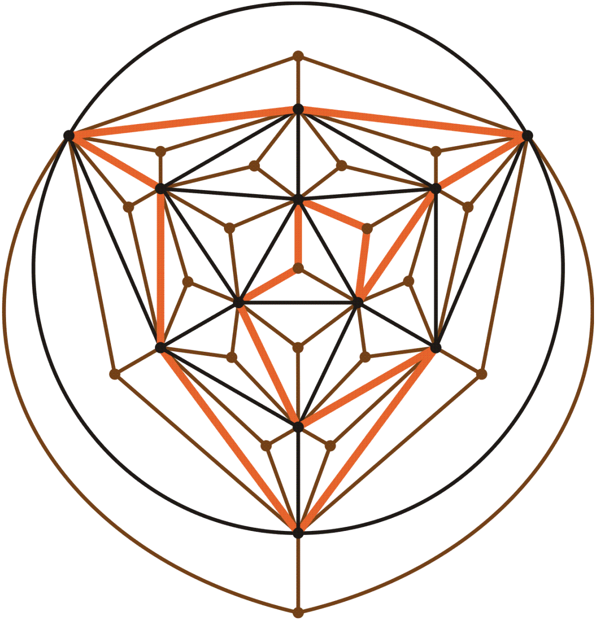

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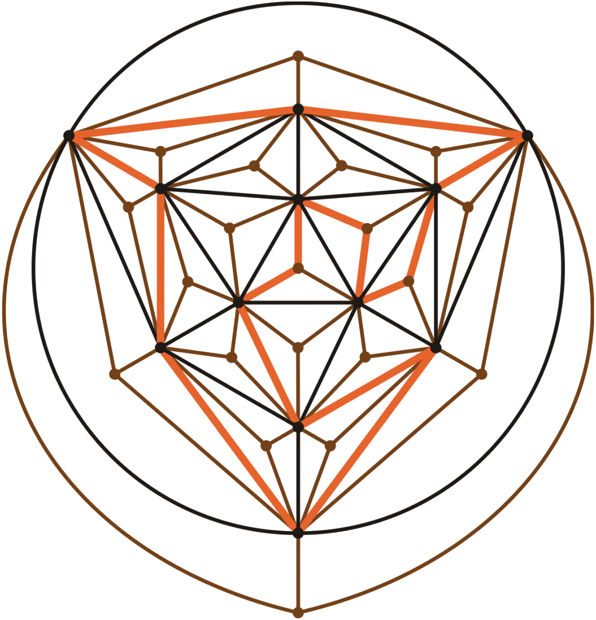

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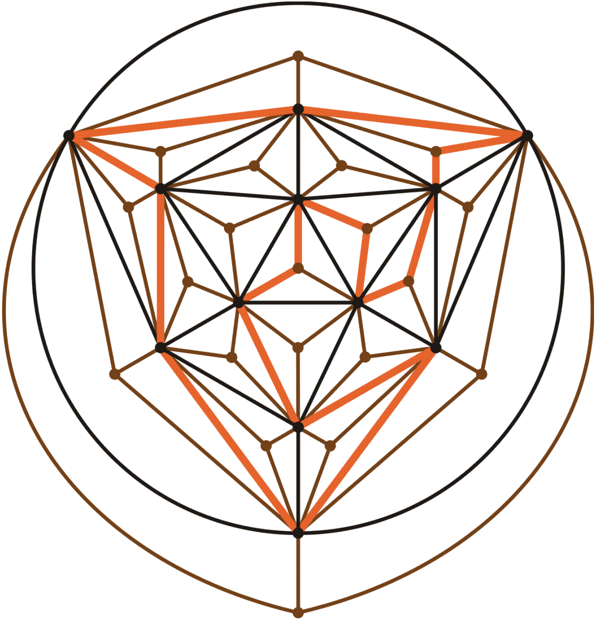

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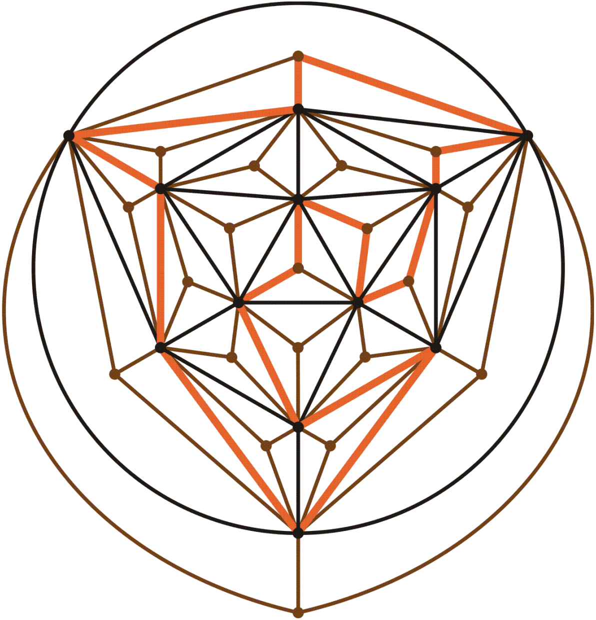

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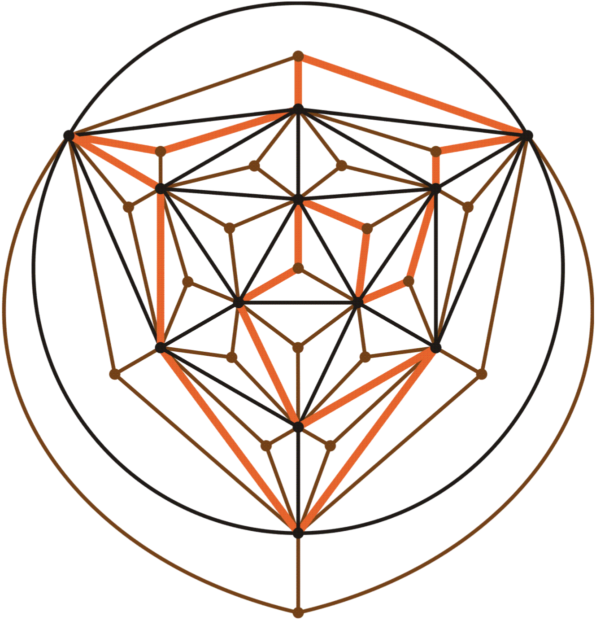

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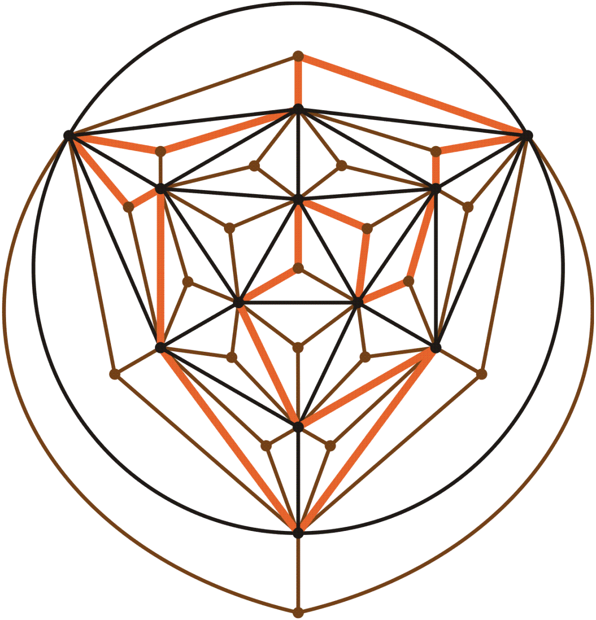

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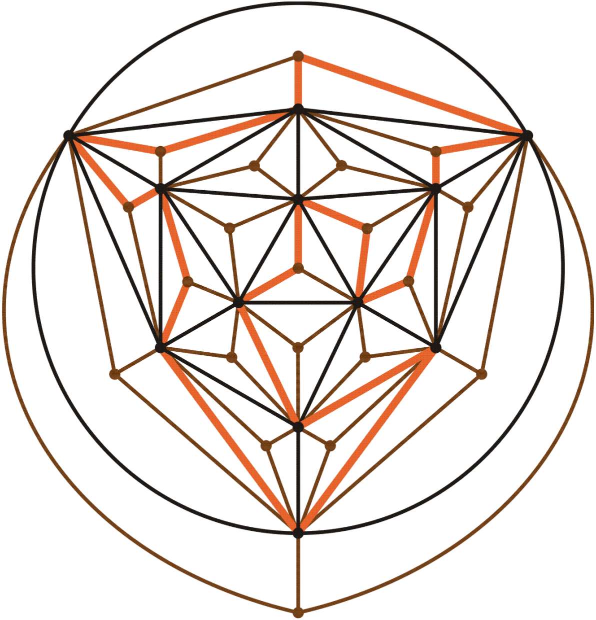

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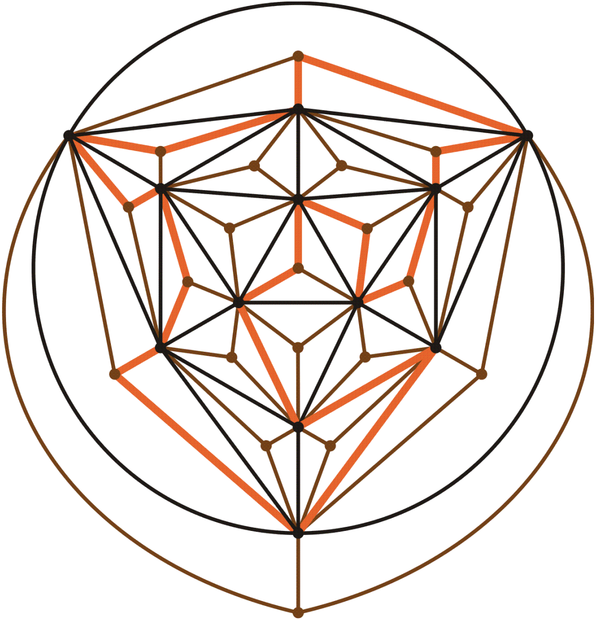

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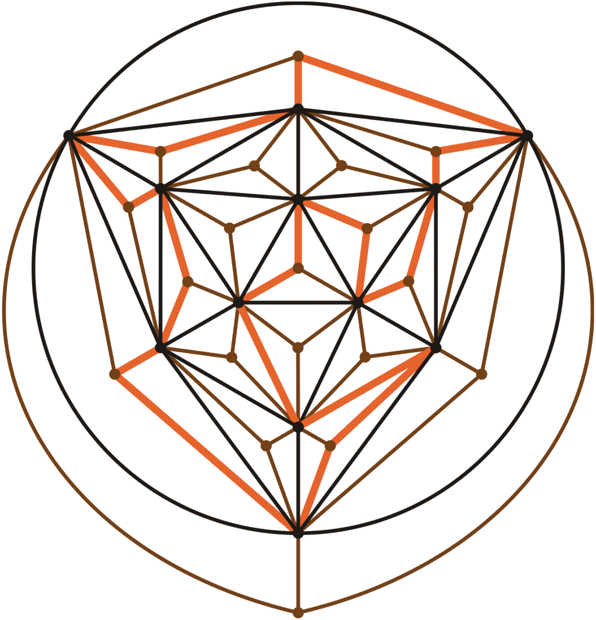

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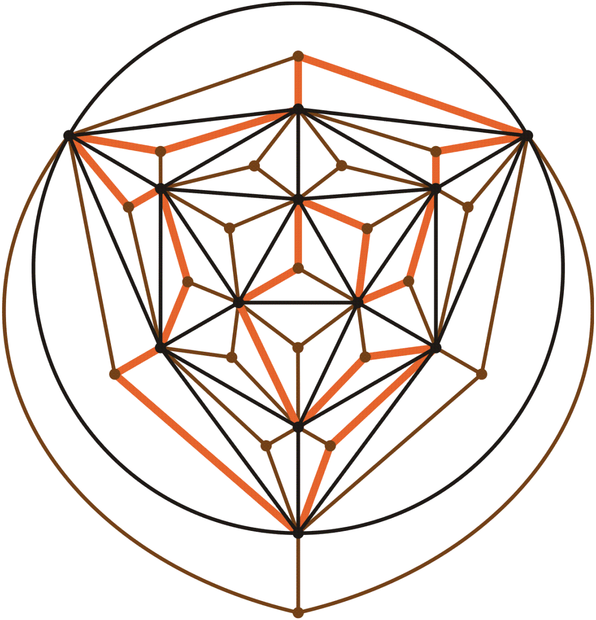

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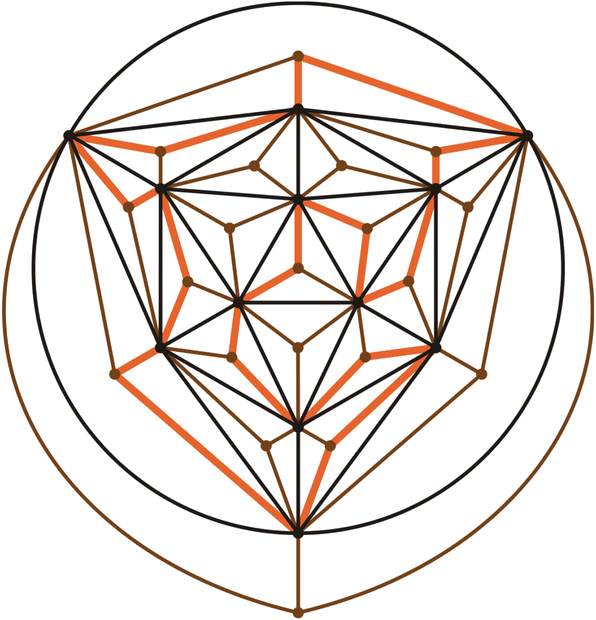

Results

Theorem (F., Harant, Mohr, Schmidt, 2019+)

For every essentially 4-connected planar graph G on n vertices,

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Theorem (F., Harant, Mohr, Schmidt, 2019+)

For every essentially 4-connected planar triangulation G on n vertices,

- $\mathrm{circ}(G) \geq \frac{2}{3}(n+4)$.

Moreover, this bound is tight.

Proof: Tutte cycle

Let G be an essentially 4-connected plane graph
and let C be a cycle of G of length at least 5.

Tutte cycle

- A cycle C of G is a *Tutte cycle*
  if $V(G) \setminus V(C)$ is an independent set of vertices of degree 3.

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extendable edge

An edge $xy \in E(C)$ is *extendable*
if there is a common neighbour $z \notin V(C)$ of x and y.

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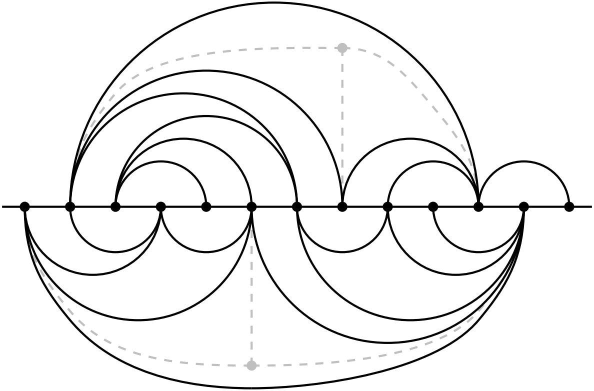
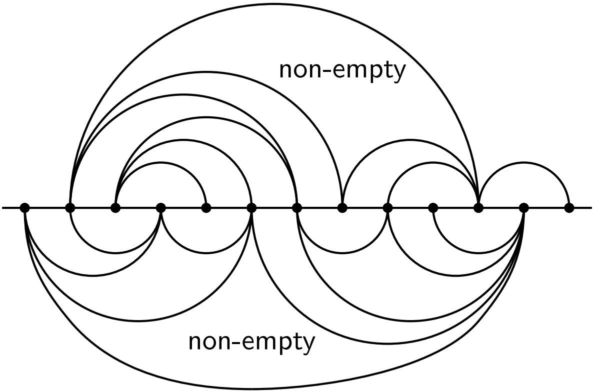
- G contains a Tutte cycle of length at least 5
- let C be a longest Tutte cycle of G
- C has no extendable edge

Proof: Tutte cycle with chords



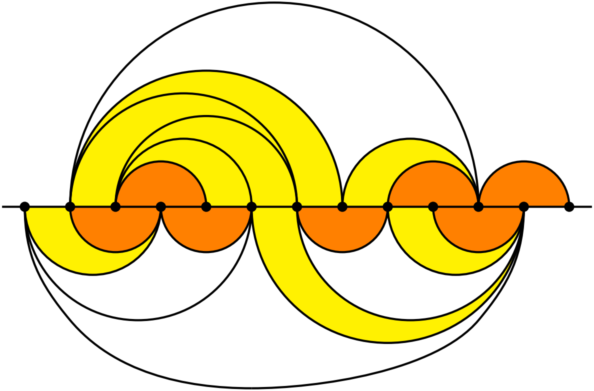
- let $H = G[V(C)]$
- H is a plane triangulation and C is a hamiltonian cycle of H

Proof: empty faces



- a face of H is *empty* if it is also a face of G
- F_0 is the set of all empty faces of H; $f_0 = |F_0|$

Proof: j-faces



- a *j-face* of H is incident with exactly j edges of $E(C)$
- each 2-face and each 1-face of H is empty

Proof: number of empty faces

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Claim

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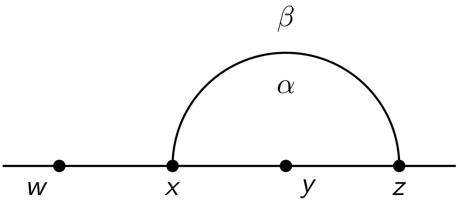
Lemma

Let $[w, x, y, z]$ be a subpath of C, let $\alpha = [x, y, z]$ be a 2-face of H_1 and let $\beta \neq \alpha$ be the face of H incident with xz. If $\varphi = [w, x, y]$ a 2-face of H_2 then β is an empty face.



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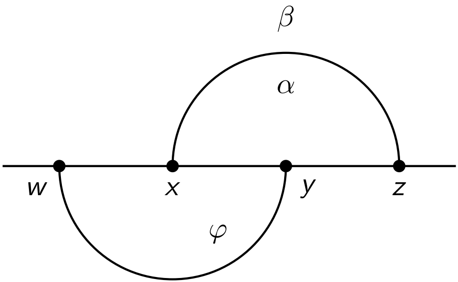

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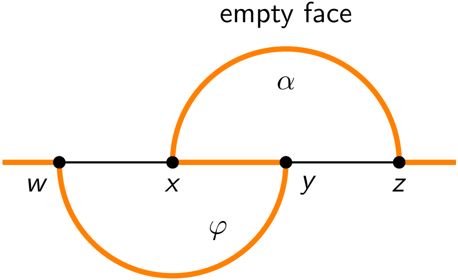

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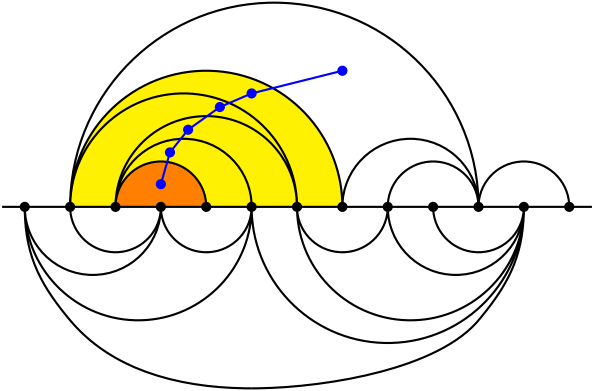

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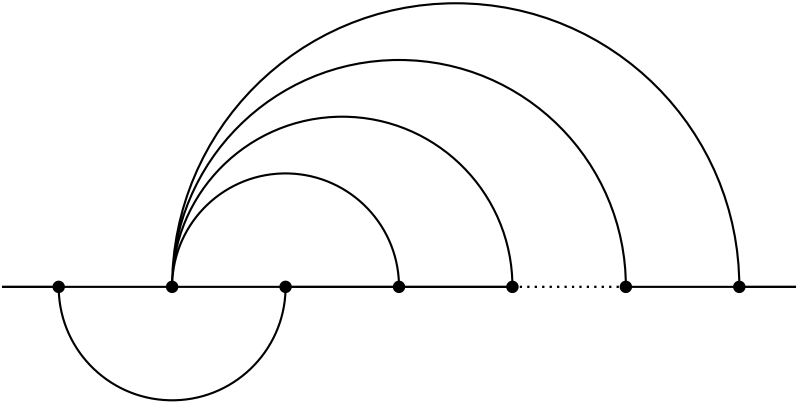

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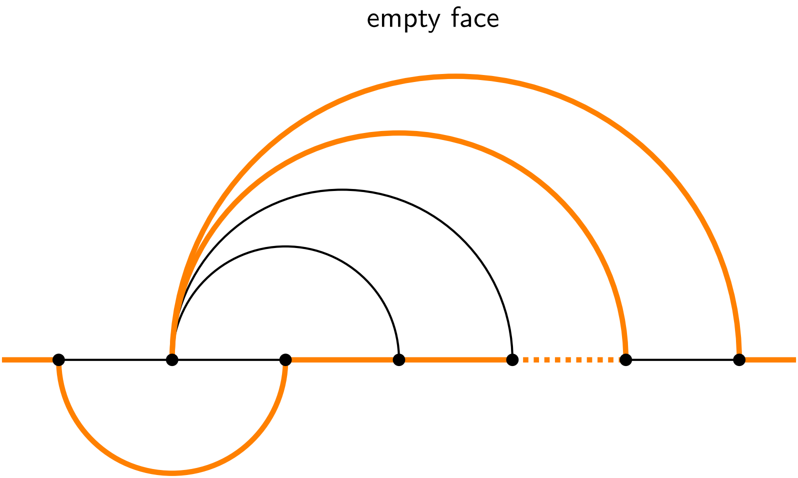

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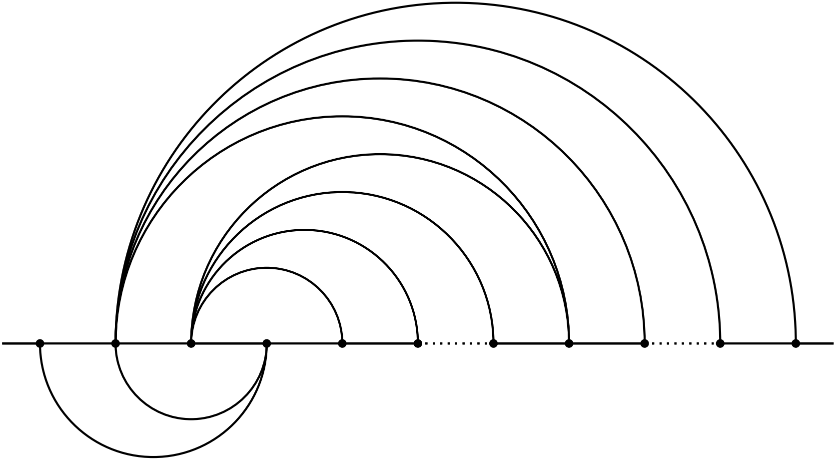

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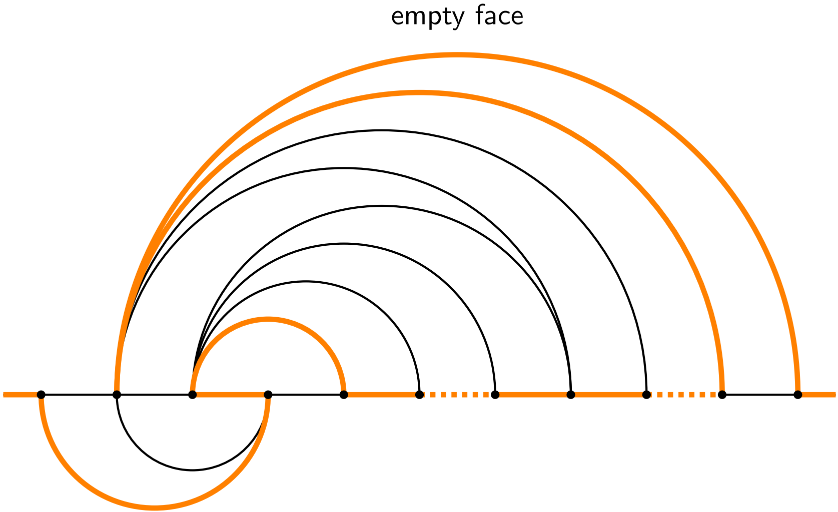

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