Aalto University
School of Science

Using Tutte Paths for Finding Long Cycles in Planar Graphs

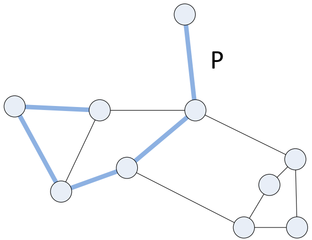
Andreas Schmid

August 12, 2019

Overview

But first... P-Bridges

For a path *P* in *G*, a *P-bridge J* is



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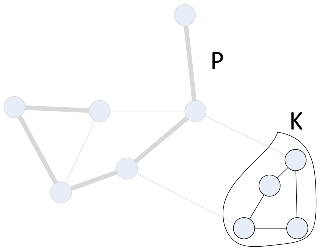
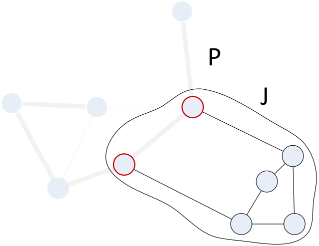


But first... P-Bridges

For a path P in G, a *P-bridge* J is

- a component K of $G \setminus V(P)$
- **plus** the edges connecting K to P and their end-vertices (**attachments**).



Tutte paths - Definition

Let G be a 2-connected plane graph with outer face C_G. Then a path T is called a **Tutte path** if:

- every T-bridge has at most **three** attachments and
- any T-bridge containing an edge of C_G has exactly **two** attachments **(outer bridges)**.

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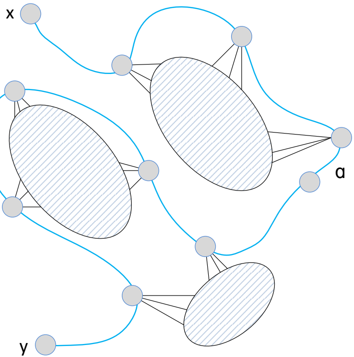
Given endvertices x and y and intermediate edge α, let $x - \alpha - y$ **path** denote a Tutte path from x to y through α.

Tutte Paths - Example



Tutte paths! What are they good for?

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Often used to show Hamiltonicity of a graph class:

- 4-connected planar graphs [Tutte56].
- 4-connected projective planar graphs [Thomas94].
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Or to make even stronger statements:

- 4-connected planar graphs are Hamiltonian-connected [Thomassen83].
- 4-connected planar graphs are 2-edge Hamiltonian-connected [Ozeki2014].
- 4-connected projective planar graphs are Hamiltonian-connected [Kawarabayaschi2015].

Using Tutte paths right!

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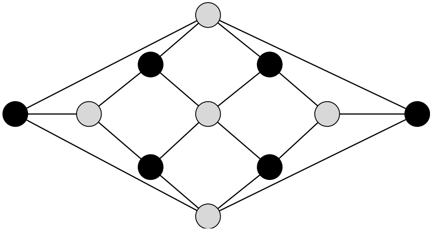

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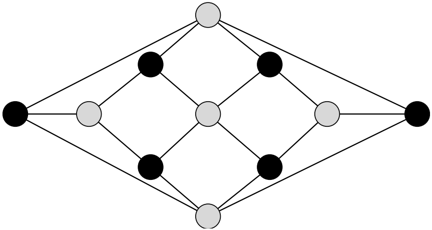

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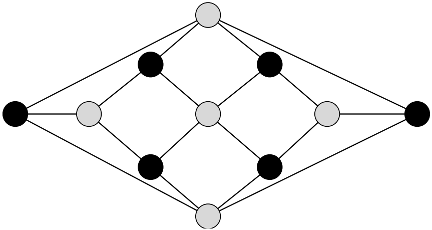

- If G has no internal 3-separator [Thomassen83].
- If G has at most three 3-separators [Brinkmann2016].

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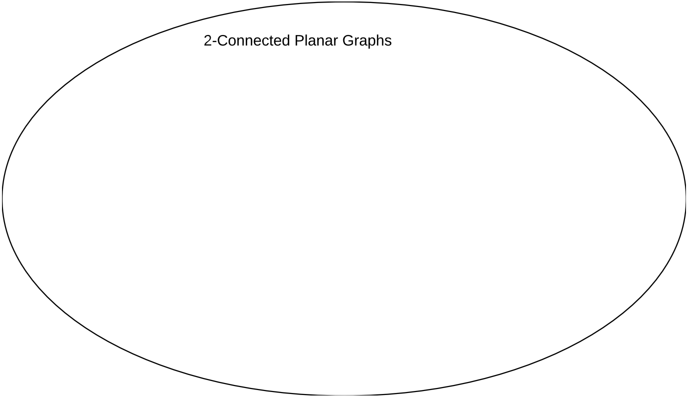

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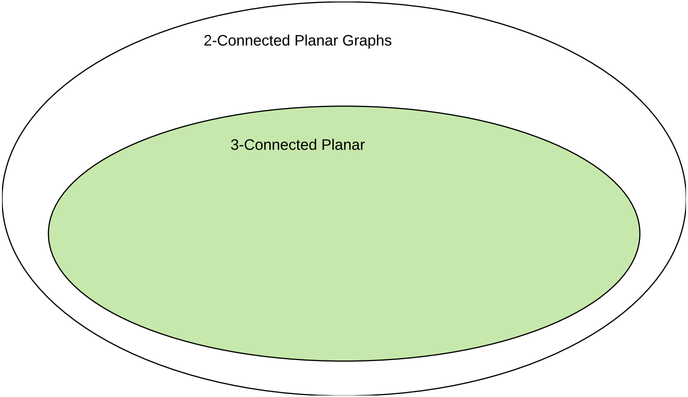

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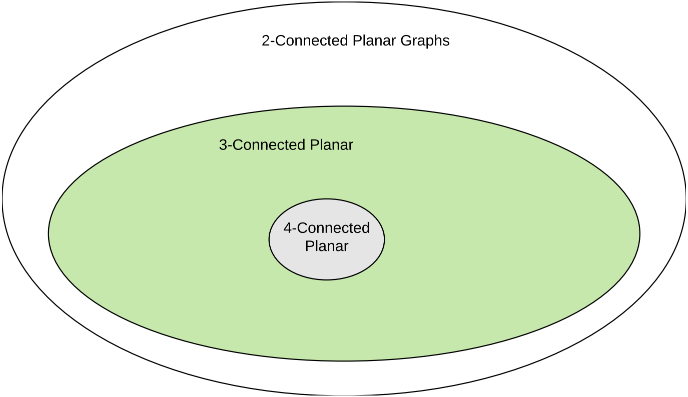

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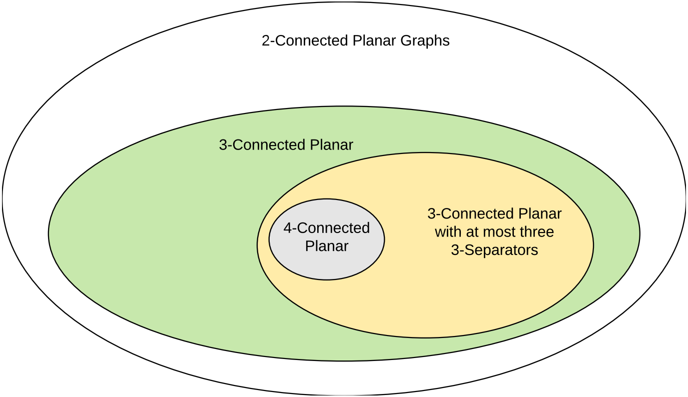

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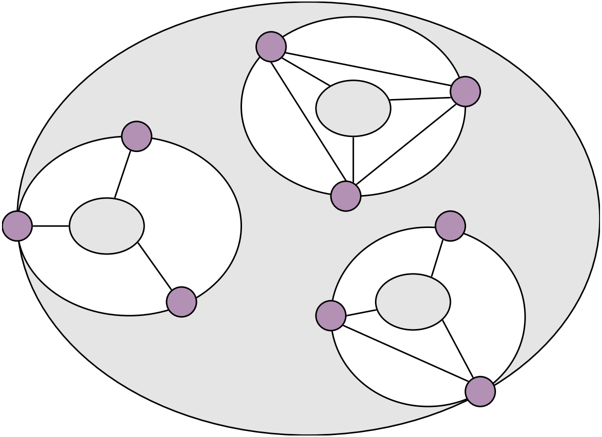
- A cycle of length at least $\frac{5(n+2)}{8}$ in essentially 4-connected planar graphs [Fabrici18].
- Every circuit graph contains a closed 2-walk [Gao95].
- Circuit graphs always have a 3-tree with few degree three vertices [Nakamoto2009].

Graphs with at most three 3-separators



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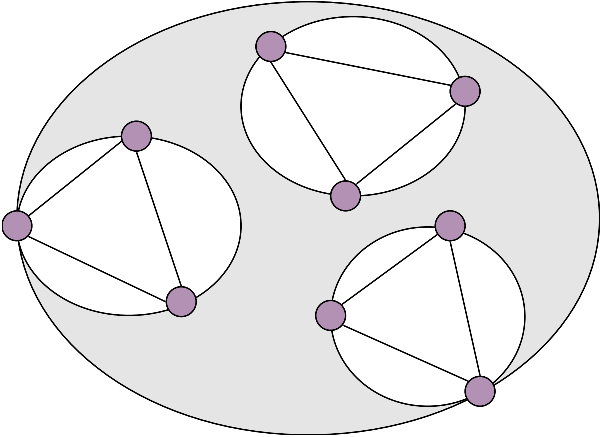

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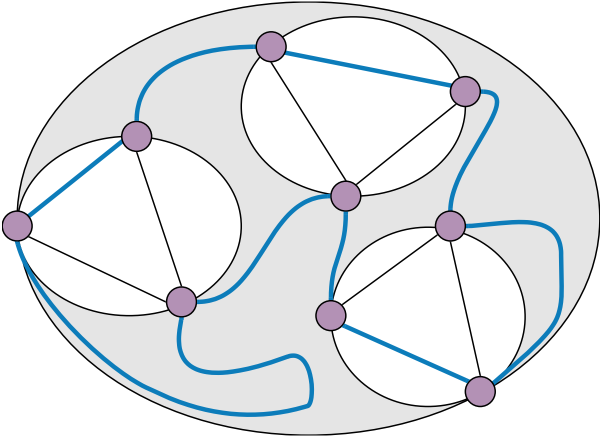

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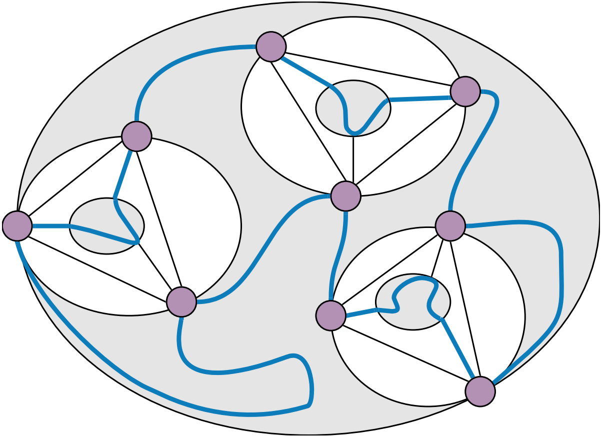

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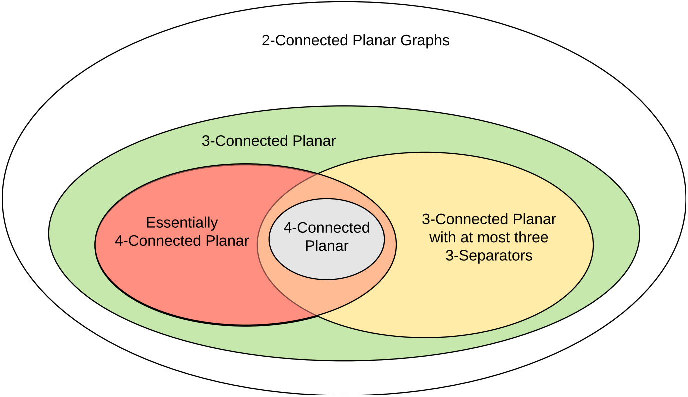

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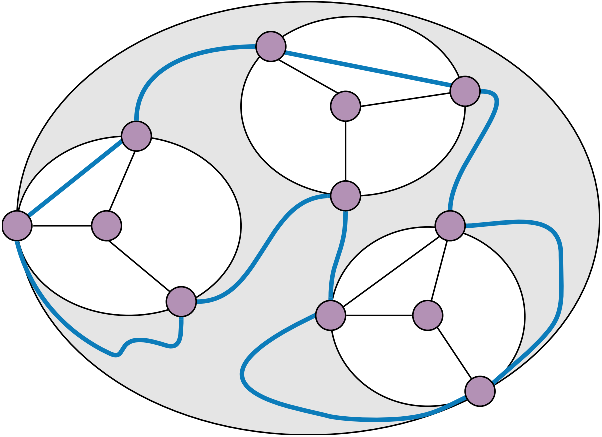

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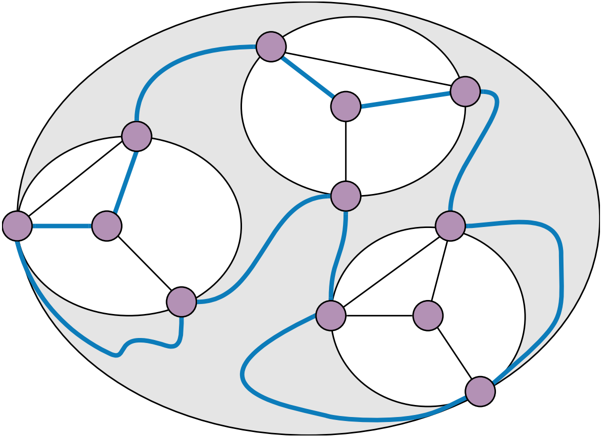
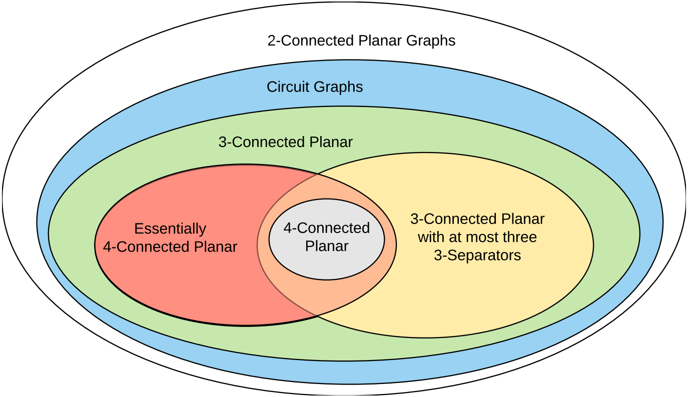


Closed 2-Walks in Circuit Graphs



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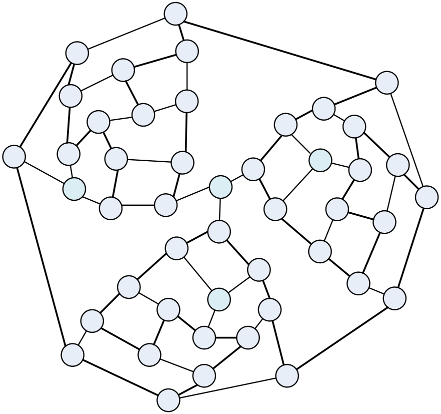

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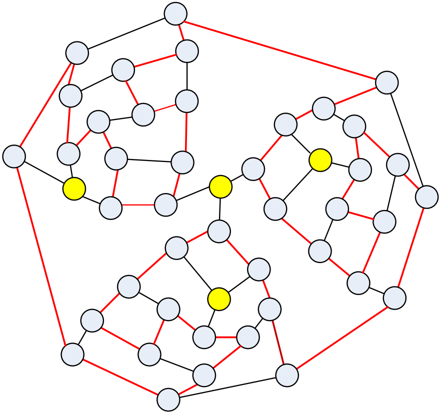

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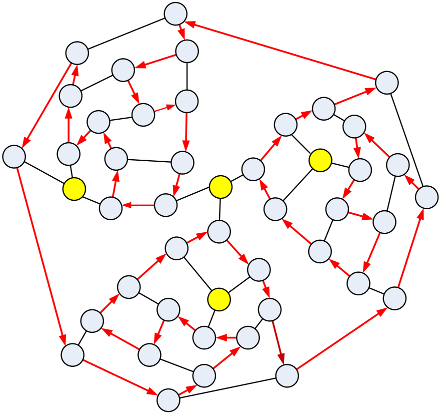

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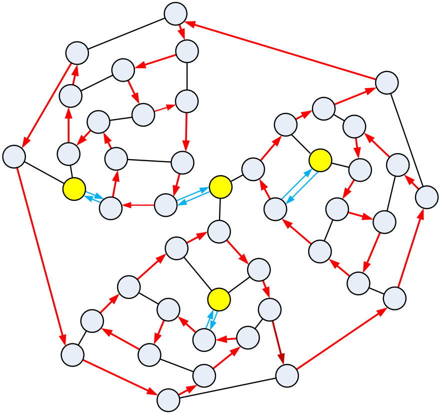
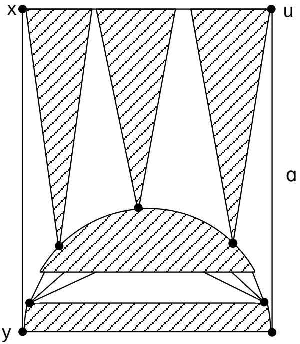


Algorithmic Difficulty in their Proofs



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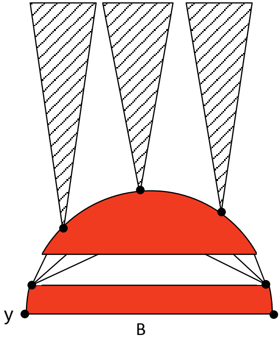

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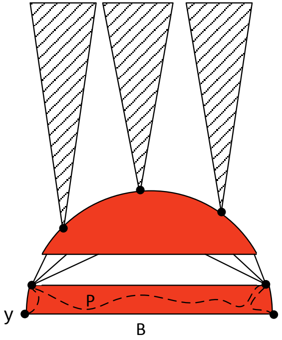

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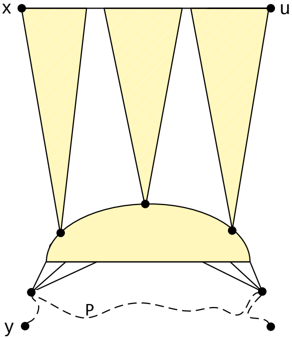

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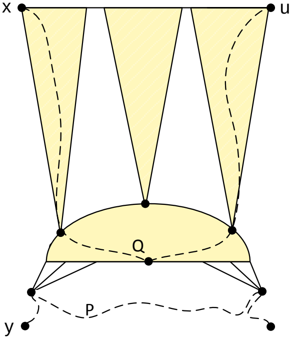

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- Construct subgraph F.
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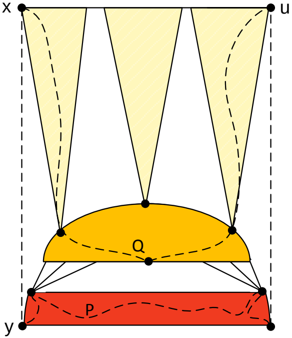

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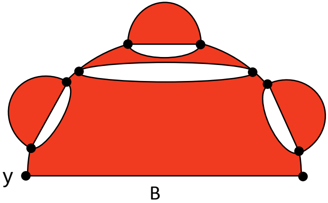

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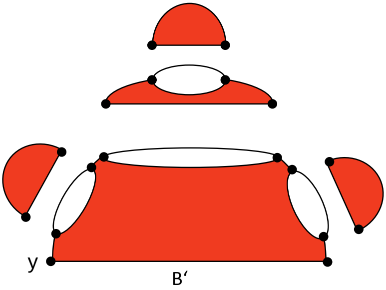

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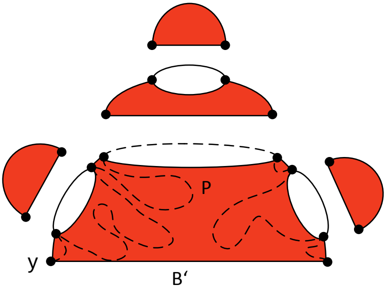

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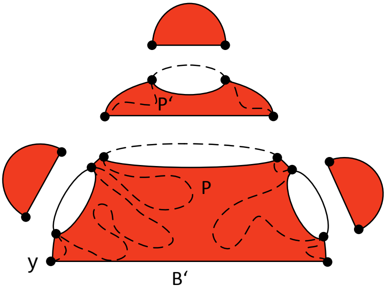

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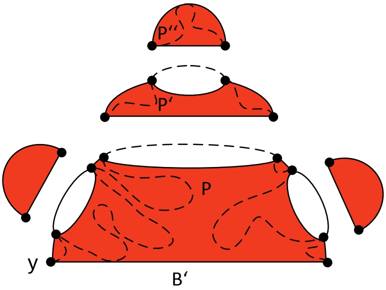

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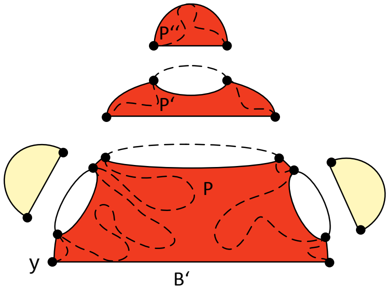

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- Merge the resulting path with paths from other blocks.

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Therefore:

Theorem

Let G be a 2-connected plane graph and let $x, y \in V(G)$ and $\alpha \in E(G)$. A $x - \alpha - y$ path in G can be found in $O(n^2)$ time.

Tutte Paths in Linear Time

In ICALP'19, Biedl and Kindermann gave the first linear time algorithm

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- Based on decomposing the graph into 3-connected subgraphs.
- Both end-vertices have to lie on the outer face.
- The authors say this cannot be generalized for arbitrary end-vertex position.

Future Work

- Find more applications...
- Improve the running time
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Questions?