Component factors of simple edge-chromatic critical graphs

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August 12, 2019

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Theorem [Vizing (1965)]

Every class 2 graph has an edge-chromatic critical subgraph.

Examples

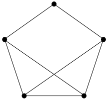

Figure: K_4 with a subdivided edge

A graph G is k-overfull if $|V(G)|$ is odd, $\Delta(G) \leq k$ and $\frac{|E(G)|}{\lfloor \frac{1}{2}|V(G)| \rfloor} > k$.

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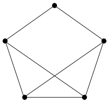

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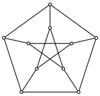

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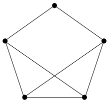

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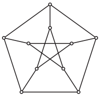

Figure: Petersen graph

Vizing's Adjacency Lemma (1964)

Let G be a critical graph. If $e = xy \in E(G)$, then at least $\Delta(G) - d_G(y) + 1$ vertices in $N(x) \setminus \{y\}$ have degree $\Delta(G)$.

Conjectures

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For all $k \geq 2$: every edge-chromatic k-critical graph has odd order. [Beineke, Wilson (1973), Jakobsen (1974)]

Conjectures

Conjecture: False

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Disproved by: Goldberg (1981) for $k = 3$; Chetwynd/Fiol (1983) for $k = 4$; Grünewald, ES (1999) for all k.

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Conjecture:

1. If G is an edge-chromatic critical graph of even order, then G has a 1-factor. [Fiorini, Wilson (1977)]
2. If G is an edge-chromatic critical graph of odd order and v is a vertex of minimum degree in G, then $G - v$ has a 1-factor. [Chetwynd, Yap (1983)]

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Vizing's 2-factor conjecture

2-Factor Conjecture [Vizing 1968]

Every edge-chromatic critical graph has a 2-factor.

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Every edge-chromatic critical graph has a 2-factor.

Results:

- trivial for 2-critical graphs.
- True for overfull graphs. [Grünewald, ES (2004)]
- If $\Delta(G) \geq \frac{6}{7}|V(G)|$, then G is Hamiltonian and thus has a 2-factor. [Lou, Zhao (2013)]

Vizing's 2-factor conjecture

Theorem [Bej, ES (2017)]

For $k \geq 3$, the following statements are equivalent:

1. Every k-critical graph has a 2-factor.
2. Every k-critical graph of even order has a 2-factor.
3. Every k-critical graph of odd order has a 2-factor.
4. Every k-critical graph G with $\delta(G) = k - 1$ has a 2-factor.
5. Every k-critical graph G with $\delta(G) = 2$ has a 2-factor.
6. For every k-critical graph G with a divalent vertex v: $G - v$ has a 2-factor.

Vizing's Independence Number Conjecture

Independence Number Conjecture [Vizing (1968)]

If G is an edge-chromatic critical graph, then $\alpha(G) \leq \frac{1}{2}|V(G)|$; i.e. every independent set in G contains at most half of the vertices of G.

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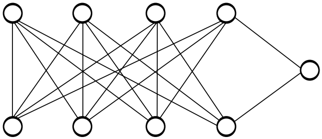

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Results:

Let G be an edge-chromatic critical graph, then

- $\alpha(G) \leq \frac{1}{2}|V(G)|$ if $|V(G)| \leq 2\Delta(G)$. [Luo and Zhao (2006)]
- $\alpha(G) < \frac{2}{3}|V(G)|$ [Brinkmann et al (2000)]; improved to

  $\alpha(G) < \frac{3}{5}|V(G)|$. [Woodall (2010)].

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Theorem [ES (2018)]

For every $\epsilon > 0$, there is a set $\mathcal{C}_\epsilon$ of edge-chromatic critical graphs such that

1. Vizing's Independence Number Conjecture is eqivalent to its restriction on $\mathcal{C}_\epsilon$, and
2. if $G \in \mathcal{C}_\epsilon$, then $\alpha(G) < (\frac{1}{2} + \epsilon)|V(G)|$.

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Q: Does every edge-chromatic critical graph have a [1,2]-factor? Yes

Theorem [Klopp, ES (2019)]

Every edge-chromatic critical graph has a [1,2]-factor.

[1,2]-factor theorem: proof idea

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For a set $S \subseteq V(G)$ let $iso(G - S)$ be the number of isolated vertices in $G - S$.
i.e. Woodall's results says $iso(G - S) \leq \frac{3}{2}|S|$ if $G - S$ is a maximum independent set of G.

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Theorem [Klopp, ES (2019)]

If G is an edge-chromatic critical graph and $S \subseteq V(G)$, then

$$\mathrm{iso}(G - S) < \left(\frac{3}{2} - \frac{1}{\Delta(G)}\right)|S|.$$

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Theorem [Akiyama, Era (1980)]

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Theorem [Klopp, ES (2019)]

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Analyzing $[1,2]$-factors

A $\{K_{1,1}, \ldots, K_{1,t}, C_m : m \geq 3\}$-factor of G is called a star-cycle factor.

Clearly, every $[1,2]$-factor can be decomposed into a $\{K_{1,1}, K_{1,2}, C_m : m \geq 3\}$-factor.
We are interested in the minimum number of $K_{1,2}$-components in such a factor.
Why?
see next slide

[1,2]-factors and fractional matchings

What can be said about component factors of graphs if $\text{iso}(G - S) \leq c|S|$ and $1 \leq c < 2$; in particular for $c = 3/2$?

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A fractional matching of G is a function $f : E(G) \to [0, 1]$ such that $\sum_{e \in E_G(v)} f(e) \leq 1$ for all $v \in V(G)$. The fractional matching number $\mu_f(G)$ is

$$\sup\{\sum_{e \in E(G)} f(e) : f \text{ is a fractional matching of } G\}.$$

Clearly, $\mu_f(G) \leq \frac{1}{2}|V(G)|$ and if $\sum_{e \in E(G)} f(e) = \frac{1}{2}|V(G)|$, then f is called a fractional perfect matching.

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Theorem [Scheinerman, Ullman (1997)]

A graph G has a fractional perfect matching if and only if $\mathrm{iso}(G - S) \leq |S|$ for all $S \subseteq V(G)$.

Star-cycle factors and fractional matchings

If F is a star-cycle factor of G, then t_i^F denotes the number of $K_{1,i}$-components of F and let $I(G) = \min\{\sum_{i=1}^{\infty}(i-1)t_i^F : F \text{ is a star-cycle factor of } G\}$.

Theorem [Klopp, ES (2019)]

Let G be a graph, $n \geq 0$ be an integer and λ be the minimum integer such that $\text{iso}(G - S) \leq \lambda|S|$ for all $S \subseteq V(G)$.

If $\mu_f(G) = \frac{1}{2}(|V(G)| - n)$, then $\lambda \leq \lceil \frac{n}{\delta(G)} \rceil + 1$ and G has a $\{K_{1,1}, \ldots, K_{1,\lambda}, C_m : m \geq 3\}$-factor F, such that $I(G) = \sum_{i=1}^{\lambda}(i-1)t_i^F = n$.

Furthermore, the $K_{1,j}$-components are induced subgraphs and their center vertices are in A and their leaves are isolated vertices in D (in the Gallai-Edmonds decomposition (D, A, C) of G).

Approximating the 2-factor conjecture: [1,2]-factors and fractional matchings

Corollary [Klopp, ES (2019)]

Let G be a graph, that has a $\{K_{1,1}, K_{1,2}, C_m : m \geq 3\}$-factor and let n be a natural number. Then, $\min(G, K_{1,2}) = n$ if and only if $\mu_f(G) = \frac{1}{2}(|V(G)| - n)$.

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Corollary [Klopp, ES (2019)]

Let G be a graph and let n, m be integers with $0 < n \leq m \leq 2n$. If $\text{iso}(G - S) \leq \frac{m}{n}|S|$ for all subsets $S \subseteq V(G)$, then

(i) $\min(G, K_{1,2}) \leq \frac{m-n}{m+n}|V(G)|$,

(ii) $\alpha(G) \leq \frac{m}{m+n}|V(G)|$.

The aforementioned Theorem of Tutte (1953) is the special case $m = n$ of this corollary.

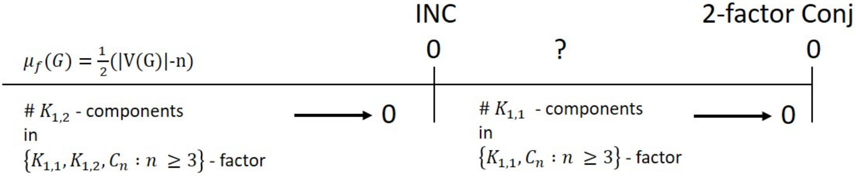
Further results

Let G be a graph with $\Delta(G) = k$. The k-deficiency $s(G)$ of G is $k|V(G)| - 2|E(G)|$. The function f with $f(e) = \frac{1}{k}$ for each $e \in E(G)$ is a fractional matching on G. Hence,

Corollary [Klopp, ES (2019)]

If G is a critical graph, then $\mu_f(G) \geq \frac{1}{2}(|V(G)| - \lfloor \frac{s(G)}{k} \rfloor)$, and therefore,
$\min(G, K_{1,2}) \leq \lfloor \frac{s(G)}{k} \rfloor)$, and $\alpha(G) \leq \frac{1}{2}(|V(G)| + \lfloor \frac{s(G)}{k} \rfloor)$.

Summary



Figure

Conjecture / Question

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2-factor Conjecture $\Rightarrow$ Fractional Perfect Matching Conjecture $\Rightarrow$ Independence Number Conjecture

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Question:

Let $k \geq 3$ and G be a k-critical graph. If G does not have a 1-factor, then $\mu_f(G) > \mu(G)$.

Thank you