Large independent sets in triangle-free subcubic graphs

Wouter Cames van Batenburg

Jan Goedgebeur

Gwenaël Joret

G graph on n vertices

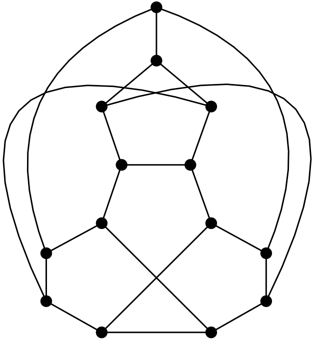
Independence number α

'Subcubic' = maximum degree at most 3

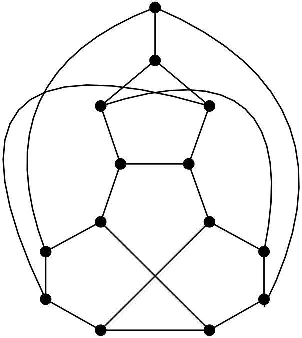
How large is α in triangle-free subcubic graphs?

Staton '79 If G subcubic and triangle-free then $\alpha \geqslant \frac{5}{14} n$

Only two tight examples among connected graphs



$n = 14 \quad \alpha = 5$

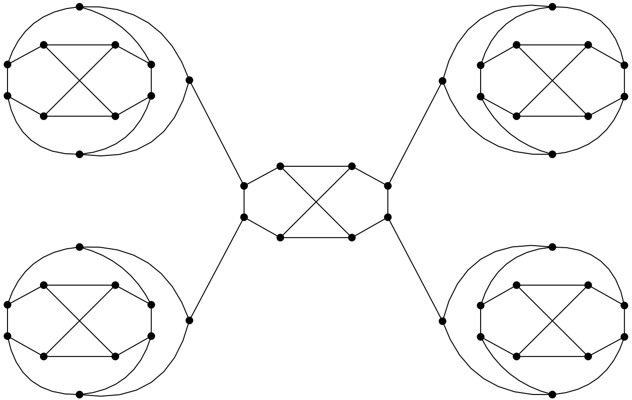


$n = 14 \quad \alpha = 5$

Fraughnaugh & Locke '95

If G subcubic, triangle-free, and connected then $\alpha \geqslant \frac{11}{30}n - \frac{2}{15}$

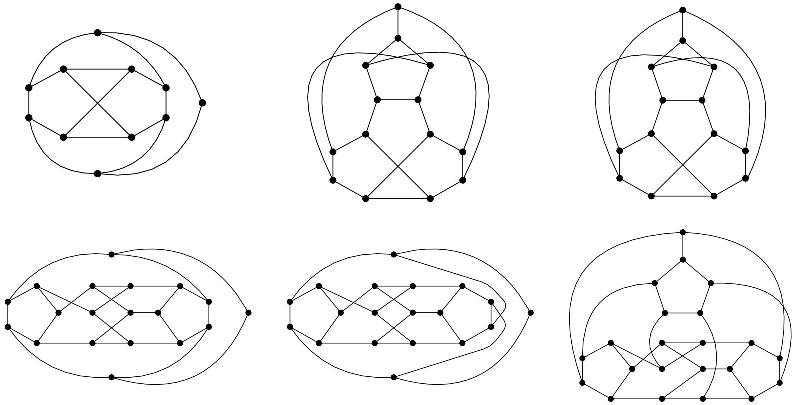
Essentially tight:



$\alpha = \frac{11}{30}n - \frac{1}{15}$

Conjecture (Locke '86)
If G subcubic, triangle-free, and 2-connected then $\alpha \geqslant \frac{3}{8}n$, except for finitely many graphs

6 exceptions (Bajnok & Brinkmann '95):



Conjecture (Fraughnaugh & Locke / Bajnok & Brinkmann '95)
If G subcubic, triangle-free, 2-connected, and G not one of the six exceptional graphs, then $\alpha \geqslant \frac{3}{8}n$

Conjecture (Albertson, Bollobas, Tucker '76)
If G subcubic, triangle-free, and planar then $\alpha \geqslant \frac{3}{8}n$

Conjecture (Fraughnaugh & Locke '95)
If G subcubic, triangle-free, and G contains none of the six exceptional graphs as subgraph then $\alpha \geqslant \frac{3}{8}n$

Conjecture (Fraughnaugh & Locke / Bajnok & Brinkmann '95)
If G subcubic, triangle-free, 2-connected, and G not one of the six exceptional graphs, then $\alpha \geqslant \frac{3}{8}n$

Heckman & Thomas '06 (conjectured by Albertson-Bollobas-Tucker '76)
If G subcubic, triangle-free, and planar then $\alpha \geqslant \frac{3}{8}n$

Conjecture (Fraughnaugh & Locke '95)
If G subcubic, triangle-free, and G contains none of the six exceptional graphs as subgraph then $\alpha \geqslant \frac{3}{8}n$

Main result

Cames van Batenburg, Goedgebeur, J. '19+
If G subcubic, triangle-free, and G contains none of the six exceptional graphs as subgraph then $\alpha \geqslant \frac{3}{8}n$

Enough to show the statement when

- G connected, and
- G *critical*, meaning $\alpha(G - e) > \alpha(G) \quad \forall e \in E(G)$

A sparsity measure:

$$\mu := \frac{9}{24} n_3 + \frac{10}{24} n_2 + \frac{11}{24} n_1 + \frac{12}{24} n_0 - \frac{2}{24}$$

where $n_i :=$ number of vertices of degree i

Equivalently:

$$\mu = \frac{6n - |E(G)| - 1}{12}$$

Remarks:

$\mu \geqslant \frac{3}{8} n - \frac{1}{12}$

$\left\lceil \frac{3}{8} n - \frac{1}{12} \right\rceil \geqslant \frac{3}{8} n$ because $n \in \mathbb{Z}$

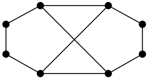
hence, to show $\alpha \geqslant \frac{3}{8} n$ it is enough to prove $\boxed{\alpha \geqslant \mu}$

Recall current assumptions:

- G subcubic and triangle-free
- G has none of the six exceptional graphs as subgraph
- G connected and critical

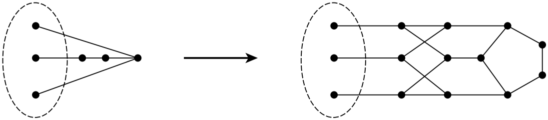
Attempt 1: Simply show that $\alpha \geqslant \mu$

Bad graphs

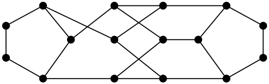


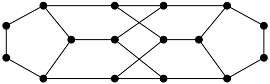
is bad

Every 8-augmentation of a bad graph is bad:



The two bad graphs on 16 vertices:





$\alpha = \mu - \frac{1}{12}$ if G bad

(however, $\alpha = \frac{3}{8}n$)

Attempt 2: Show that $\alpha \geqslant \mu$, unless G is bad

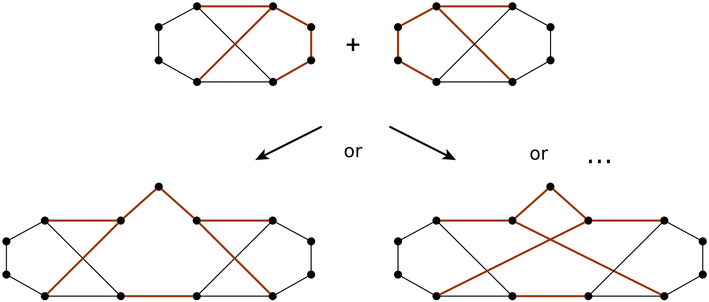
This is true

To prove this, we consider a slightly stronger statement

Dangerous graphs

C_5 is dangerous

Join of two bad graphs is dangerous:



$\alpha = \mu$ if G dangerous

Main technical theorem (CvB-G-J '19+)
Suppose

- G subcubic and triangle-free
- G has none of the six exceptional graphs as subgraph
- G connected and critical, and
- G not bad

then $\alpha \geqslant \mu$.

If moreover

- G has $\geqslant 3$ degree-2 vertices and
- G not dangerous

then $\alpha \geqslant \mu + \frac{1}{12}$.

Plan of the proof

G minimum counter-example

- G almost 3-connected: If X is a 2-cutset then $G - X$ has exactly two components, with one isomorphic to K_1 or K_2
- G has no bad subgraph
- Deal with degree-2 vertices:
  - case where neighbors have both degree 2
  - case where neighbors have both degree 3
  - case where neighbors have degree 2 and 3

$\rightarrow$ G is cubic and 3-connected

- G has no 4-cycle
- G has no 6-cycle
- G has no dangerous subgraph (in particular, no 5-cycle)

Final argument: Local structure around a shortest even cycle

Open problems

Staton '79 If G subcubic and triangle-free then $\frac{n}{\alpha} \leqslant \frac{14}{5}$

Recall: $\frac{n}{\alpha} \leqslant \chi_f$

Dvořák, Sereni, Volec '14 (conjectured by Heckman & Thomas '01)
If G subcubic and triangle-free then $\chi_f \leqslant \frac{14}{5}$

Could the upper bound on χ_f be improved if we further assume

- G connected, or
- G 2-connected, or
- G planar, or
- G has none of the 6 exceptional graphs as subgraph?