DETECTING ODD HOLES

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$G(V, E)$ GRAPH

$V \to$ VERTICES, $E \to$ EDGES

CLIQUE A SET OF PW ADJACENT VERTICES

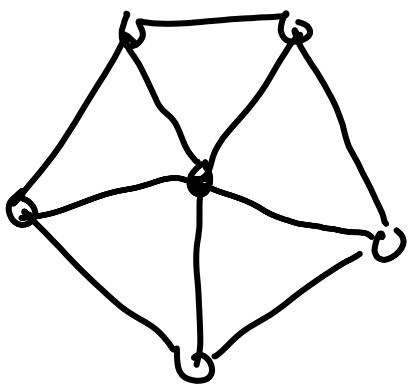
$\omega(G)$ MAX SIZE OF A CLIQUE

$\chi(G)$ CHROMATIC NUMBER

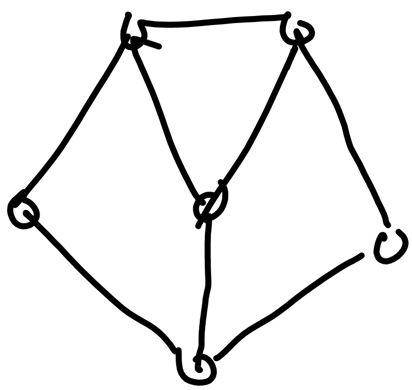
FACT $\chi(G) \geq \omega(G)$

H IS AN INDUCED SUBGRAPH OF G IF

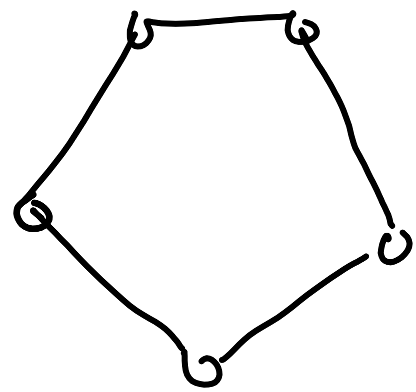
- $V(H) \subseteq V(G)$
- $uv \in E(H)$ IFF $uv \in E(G)$ & $u, v \in V(H)$



G



NOT ISG OF G



ISG OF G

G IS PERFECT IF $\chi(H) = \omega(H)$ FOR EVERY INDUCED SUBGRAPH H OF G

MANY NATURAL CLASSES OF GRAPHS ARE PERFECT

WEAK PERFECT GRAPH THEOREM (LOVASZ, 1972)

G IS PERFECT IFF G^c IS PERFECT

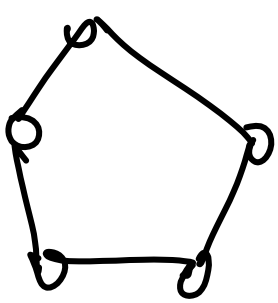
DEF: $V(G^c) = V(G)$

$uv \in E(G^c)$ IFF $uv \notin E(G)$

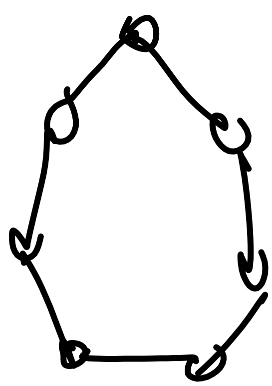
STRONG PERFECT GRAPH THEOREM

(C., ROBERTSON, SEYMOUR, THOMAS 2003)

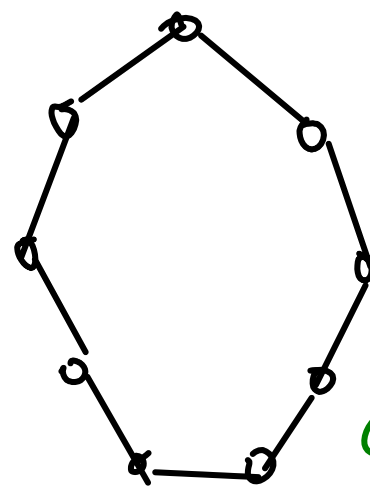
G IS PERFECT IFF NO INDUCED SUBGRAPH OF G IS ISOMORPHIC TO C_{2k+1} OR C_{2k+1}^c WITH $k \geq 2$



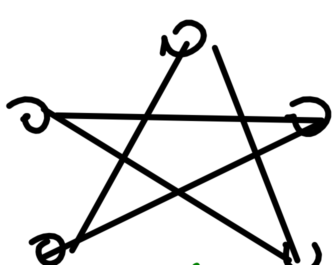
C_5



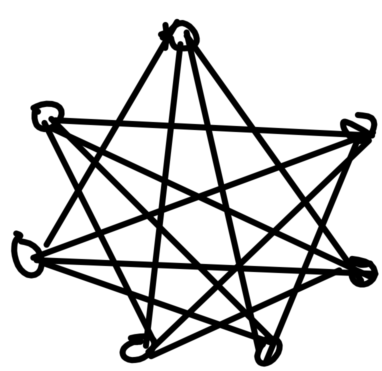
C_7



C_9



C_5^c



C_7^c

(5)

A HOLE IN G IS AN INDUCED SUBGRAPH ISOMORPHIC TO C_k WITH $k \geq 4$

ODD HOLE, EVEN HOLE

AN ANTIHOLE IN G IS AN INDUCED SUBGRAPH ISOMORPHIC TO C_k^c WITH $k \geq 4$

ODD ANTIHOLE, EVEN ANTIHOLE

G IS BERGE IF G HAS NO ODD HOLE AND NO ODD ANTIHOLE

SPGT BERGE = PERFECT

THM (GROTSCHEL, LOVASZ, SCHRIJVER)

$\chi(G)$, $\omega(G)$, $\alpha(G)$

CAN BE CALCULATED IN POLYNOMIAL TIME IF G IS PERFECT

QUESTION:

CAN ONE TEST IN POLYNOMIAL TIME IF A GRAPH IS PERFECT?

THM (C, CORNUEJOLS, LIU, SEYMOUR, VUSKOVIC; 2003)

ONE CAN TEST IN TIME
($O(|V(G)|^9)$) IF G IS BERGE

COR TESTING PEFECTION IS POLYNOMIAL

QUESTION WHAT IS THE COMPLEXITY OF TESTING FOR ODD HOLES?

FACTS RELATED TO TESTING FOR ODD HOLES

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- TESTING FOR EVEN HOLES IS POLYNOMIAL (CONFORTI, CORNUEJOLS, KAPOOR & VUSKOVIC; CHUDNOVSKY, KAWARABAYASHI, SEYMOUR; DA SILVA, VUSKOVIC)

- IT IS NP-COMPLETE TO TEST IF AN INPUT GRAPH CONTAINS AN ODD HOLE THROUGH A GIVEN VERTEX (BIENSTOCK)

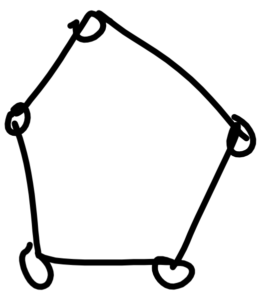
THM (C., SCOTT, SEYMOUR, SPIRKL)
ONE CAN TEST IN TIME
$O(|V(G)|^9)$ IF G CONTAINS
AN ODD HOLE

A STRONGER THEOREM
(C., SCOTT, SEYMOUR)
$\forall \ell$ ONE CAN TEST IN TIME
$O(|V(G)|^{O(\ell)})$ IF G CONTAINS
AN ODD HOLE OF LENGTH $\geq \ell$

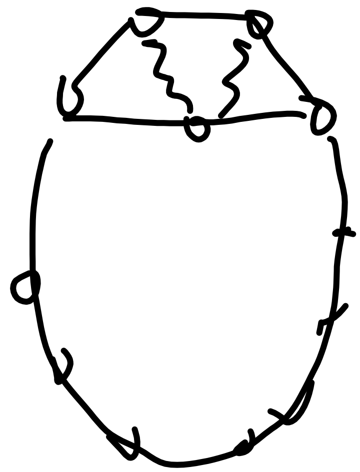
THREE STEPS

1. EASILY DETECTABLE CONFIGURATIONS
2. CLEANING
3. SHORTEST PATHS DETECTOR FOR A CLEAN SHORTEST ODD HOLE

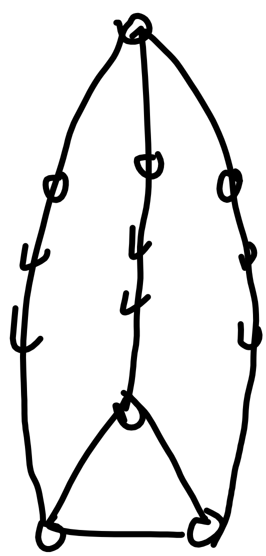
EASILY DETECTABLE CONFIGURATIONS



C5



JEWEL

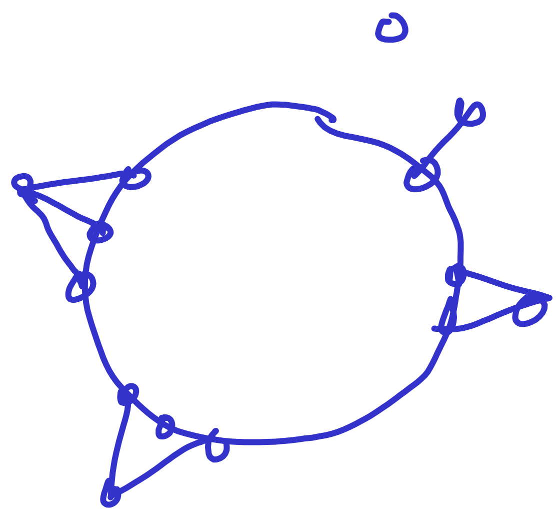


PYRAMID

- CAN TEST IN POLY TIME
- IF YES, THEN G CONTANS AN ODD HOLE

LET C BE A SHORTEST ODD HOLE IN G

$v \notin V(C)$ IS C-MINOR IF $N(v) \cap V(C)$ IS CONTAINED IN A 3-VERTEX PATH OF C

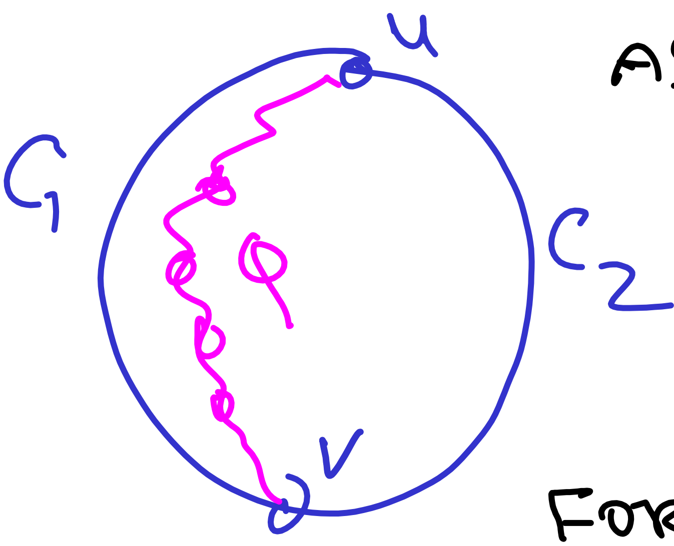


$v \notin V(C)$ IS C-MAJOR IF v IS NOT C-MINOR $M(C)$.

C IS CLEAN IF NO VERTEX IS C-MAJOR

SHORTEST PATHS DETECTOR

THM LET G BE A GRAPH WITH NO JEWEL AND NO PYRAMID, LET C BE A CLEAN SHORTEST ODD HOLE IN G. LET $u,v \in V(C)$



ASSUME $|V(C_1)| < |V(C_2)|$.

THEN:

C_1 IS A SHORTEST u-v PATH IN G, AND FOR EVERY SHORTEST u-v PATH Q IN G, u-Q-v-C_2-u IS A CLEAN SHORTEST ODD HOLE IN G

SHORTEST PATHS ALG

INPUT G

OUTPUT A DETERMINATION IF G CONTAINS A CLEAN SHORTEST ODD HOLE

- FOR EVERY $v_1, v_2, v_3 \in V(G)$
- COMPUTE P_1 : SHORTEST $v_2 - v_3$ PATH IN $G \setminus v_1$
- COMPUTE P_2, P_3
- TEST IF $v_2 - P_1 - v_3 - P_2 - v_1 - P_3 - v_2$ IS AN ODD HOLE
- IF "YES" OUTPUT "YES"

IF "NO" FOR ALL (v_1, v_2, v_3) OUTPUT "G HAS NO CLEAN SHORTEST ODD HOLE"

CLEANING

CONSTRUCT SETS

$X_1, \dots, X_k \subseteq V(G)$ S.T.

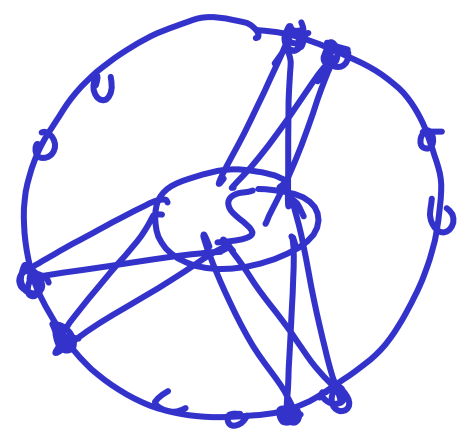
- $k = \text{poly}(|V(G)|)$
- IF C IS A SHORTEST ODD HOLE IN G, THEN $\exists i$ S.T.
  - $X_i \cap V(C) = \emptyset$
  - $M(C) \subseteq X_i$

NOW C IS A CLEAN SHORTEST ODD HOLE IN $G \setminus X_i$

RUN THE ALG FROM THE PREVIOUS SLIDE ON EACH $G \setminus X_i$

CONSTRUCTING THE SETS $X_1, \ldots, X_k$

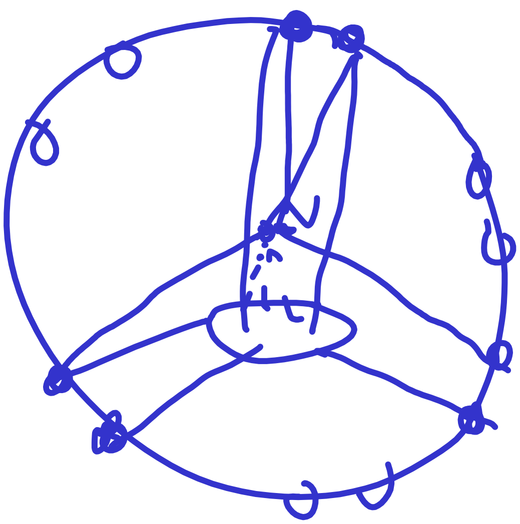
THM LET C BE A SHORTEST ODD HOLE IN G, AND LET $S \subseteq M(C)$ BE STABLE. THEN AN ODD # OF EDGES OF C IS S-COMPLETE



DEF $e=ab$ IS S-COMPLETE IF $\forall s \in S$ $as \in E(G)$ & $bs \in E(G)$

DEF $e = ab$ IS X-HEAVY (17)
IF $\forall x \in X$ $ax \in E(G)$ OR $bx \in E(G)$

THM LET C BE A SHORTEST ODD HOLE IN G, LET $v \in M(C)$ AND LET $X = M(C) \setminus N(v)$. THEN SOME EDGE OF C IS X-HEAVY



REMARK THIS GIVES AN ALG TO TEST FOR ODD HOLES IN GRAPHS WITH BOUNDED CLIQUE NUMBER

(ORIGINALLY A RESULT OF CONFORTI, CORNUÉJOLS, LIU, VUSKOVIC).

AS BEFORE

C SHORTEST ODD HOLE

$v \in M(C)$

CAN CONSTRUCT $|V(G)|^4$ SUBSETS $X_1', \dots, X_k' \subseteq V(G)$

S.T. $\exists i$ WITH

- $M(C) \setminus N(v) \subseteq X_i'$
- $V(C) \cap X_i' = \emptyset$

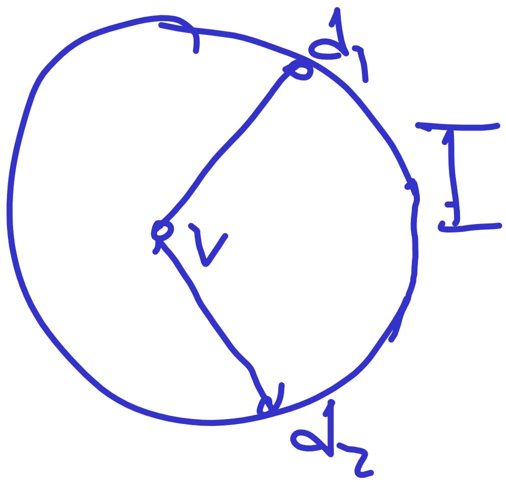
AS FOLLOWS:

$\forall\ a - b - c - d$ let

$$X_{a,b,c,d} = (N(b) \cup N(c)) \setminus \{a,d\}$$

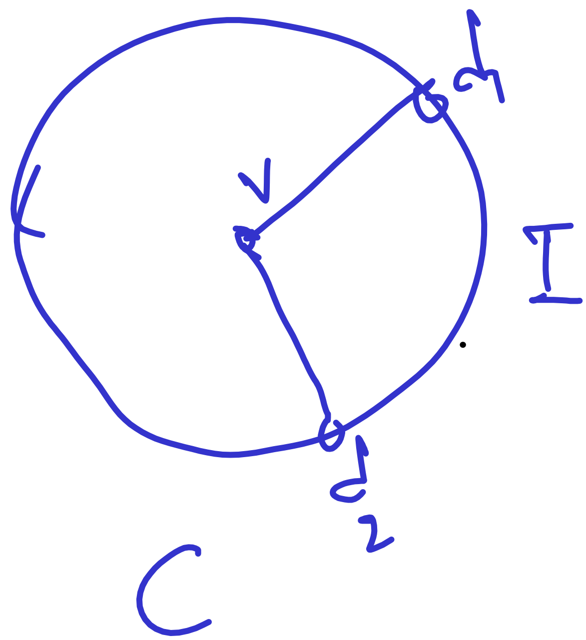
$\boxed{\text{SO WMA } M(C) \subseteq N[v]}$

- C shortest odd hole in G
- $v \in M(C)$ with max'l interval I



- Every $m \in M(C)$ either has a nbr in I^*, or $d_1 m \in E(G)$ & $d_2 m \in E(G)$
- Assume $M(C) \subseteq N[v]$
- Let Y be the set of vertices of $G \setminus N[v]$ that are in d_1-d_2 shortest paths of $G \setminus N[v]$. Then $I^* \subseteq Y$
- Let X be the set of vertices of $N(v)$ with a nbr in Y.
- $M(C) \subseteq X \cup (N(d_1) \cap N(d_2))$ and $X \cap V(C) = \{d_1, d_2\}$

FOR EVERY v, d_1, d_2 WE CAN CONSTRUCT $X_{v,d_1,d_2} \cup (N(d_1) \cap N(d_2))$ AS IN THE PREVIOUS SLIDE



FOR THE "RIGHT CHOICE" OF v, d_1, d_2

$$\left(X_{v,d_1,d_2} \cup (N(d_1) \cap N(d_2))\right) \setminus \{d_1, d_2\}$$

HAS THE REQUIRED PROPERTIES

THANK YOU !