Some Folkman Problems

existence and non-existence of
generalized Folkman numbers,
computational challenges

Stanisław Radziszowski

Department of Computer Science
Rochester Institute of Technology, NY

joint work with Xiaodong Xu and Meilian Liang

GGTW, 16–18 August 2017, Ghent

Erdős and Hajnal

Research Problem 2-5, JCT 2, p. 105, 1967

Construct a graph G which does not contain a complete hexagon such that for every coloring of the edges by two colors there is a triangle all of whose edges have the same color.

done by R.L. Graham, 1968

The proposers expect that for every cardinal m there is a graph G which contains no complete quadrilateral such that for every coloring of the edges by m colors there is a triangle all of whose edges have the same color.

proved for $m = 2$ by Folkman, 1970
generalized by Nešetřil and Rödl, 1976

50 Years of the Most Wanted Folkman Number

What is the smallest order n of a K_4-free graph
which is not a union of two triangle-free graphs?

year	lower/upper bounds	who/what
1967	any?	Erdős-Hajnal
1970	exist	Folkman
1972	10 –	Lin
1975	– 10^{10}?	Erdős offers \$100 for proof
1986	– 8×10^{11}	Frankl-Rödl, almost won
1988	– 3×10^9	Spencer, probabilistic, won \$100
1999	16 –	Piwakowski-R-Urbański, implicit
2007	19 –	R-Xu
2008	– 9697	Lu, eigenvalues
2008	– 941	Dudek-Rödl, maxcut-SDP
2012	– 100?	Graham offers \$100 for proof
2014	– 786	Lange-R-Xu, maxcut-SDP
2016	20 – 785	Bikov-Nenov / Kaufmann-Wickus-R

Folkman Graphs and Numbers

For graphs F, G, H and positive integers s, t

- $F \to (s,t)^e$ iff in every 2-coloring of the edges of F there is a monochromatic K_s in color 1 or K_t in color 2
- $F \to (G,H)^e$ iff in every 2-coloring of the edges of F there is a copy of G in color 1 or a copy of H in color 2
- variants: coloring vertices, arrowing general graphs, more colors

Edge Folkman graphs
$\mathcal{F}_e(s,t;k) = \{F \mid F \to (s,t)^e,\ K_k \not\subseteq F\}$

Edge Folkman numbers
$F_e(s,t;k) =$ the smallest order of graphs in $\mathcal{F}_e(s,t;k)$

on the previous slide we discussed $F_e(3,3;4)$

Theorem (Folkman 1970)
If $k > \max(s,t)$, then $F_e(s,t;k)$ and $F_v(s,t;k)$ exist.

Bounds from Chromatic Numbers

Set $m = 1 + \sum_{i=1}^{r}(a_i - 1)$, $M = R(a_1, \cdots, a_r)$.

Theorem (Nenov 2001, Lin 1972, others)

If $G \to (a_1, \cdots, a_r)^v$, then $\chi(G) \geq m$.
If $G \to (a_1, \cdots, a_r)^e$, then $\chi(G) \geq M$.

Special Case of Folkman Numbers

is just about graph chromatic number $\chi(G)$

Note: $G \to (2 \cdots_r 2)^v \iff \chi(G) \geq r+1$

For all $r \geq 1$, $F_v(2^r;3)$ exists and it is equal to
the smallest order of $(r+1)$-chromatic triangle-free graph.

$F_v(2^{r+1};3) \leq 2F_v(2^r;3)+1$, Mycielski construction, 1955

small cases

$F_v(2^2;3) = 5, \quad C_5$, Mycielskian, 1955

$F_v(2^3;3) = 11$, the Grötzsch graph, Mycielskian, 1955

$F_v(2^4;3) = 22$, Jensen and Royle, 1995

$F_v(2^5;3) \leq 44$, Droogendijk, 2015

$32 \leq F_v(2^5;3) \leq 40$, Goedgebeur, 2017

Generalized Folkman Problems

Arrowing and avoiding general graphs

$F_v(H_1, H_2; H) =$ smallest n for which there exists an H-free graph G of order n such that $G \to (H_1, H_2)^v$

$F_e(H_1, H_2; H) =$ smallest n for which there exists an H-free graph G of order n such that $G \to (H_1, H_2)^e$

- When H_1, H_2, H are complete graphs this is classics
- Some existence questions are discussed in the following
- Some other existence questions seem very difficult
- $F_e(K_4 - e, K_4 - e; K_4) \leq 30193$, Lu 2008

  side effect of an attack on $F_e(3, 3; 4)$

Avoiding $K_k - e$

Notation: $J_k = K_k - e$

Theorem

For every integer $k \geq 3$,

(a) *the edge Folkman number $F_e(K_{k+1}, K_{k+1}; J_{k+2})$ exists, and*

(b) *the vertex Folkman number $F_v(K_k, K_k; J_{k+1})$ exists.*

Proof:
Based on a result by Nešetřil and Rödl (1981), and on our lemma.

Challenge: compute the following

$F_v(K_3, K_4; J_5)$, perhaps doable
$F_e(K_3, K_3; J_5)$, almost hopeless
$F_e(K_3, K_3; K_4)$, 50 years intro slide, hopeless

Avoiding Books

Notation: $B_k = K_1 + K_{1,k}$, hence also $B_2 = J_4$

Theorem

The edge Folkman number $F_e(K_3, K_3; B_3)$ does not exist.

Problem

Does the edge Folkman number $F_e(K_3, K_3; B_4)$ exist?

Clearly, $F_e(K_3, K_3; B_k) = 6$ for all $k \geq 5$, hence we study B_k-free and K_4-free graphs arrowing $(3, 3)^e$.

Existence of $F_e(K_3, K_3; H)$ for small $H \supset K_4$

Notation: Graphs $\widehat{K}_{4,i} \subset K_5$ for $i \in [4]$, where $\widehat{K}_{n,s}$ is the graph obtained by connecting a new vertex v to s vertices of a K_n.

Lemma

$15 = F_e(K_3, K_3; K_5) \leq F_e(K_3, K_3; J_5) \leq F_e(K_3, K_3; K_4) \leq 785.$

(observe that $\widehat{K}_{4,4} = K_5$, $\widehat{K}_{4,3} = J_5$)

Proof: by monotonicity.

Lemma

$F_e(K_3, K_3; \widehat{K}_{4,2}) = F_e(K_3, K_3; \widehat{K}_{4,1}) = F_e(K_3, K_3; K_4).$

Proof: short and cool.

R · I · T
Department of Computer Science

Existence of $F_e(K_3, K_3; H)$ for small $H \not\supset K_4$

Theorem

The edge Folkman number $F_e(K_3, K_3; K_1 + P_4)$ does not exist.

Theorem

Let H be any connected K_4-free graph on 5-vertices containing K_3. Then the edge Folkman number $F_e(K_3, K_3; H)$ does not exist, except for two possible cases for H, namely W_5 and $\overline{P_2 \cup P_3}$.

Proofs: some work but not that hard.

Existence of $F_e(K_3, K_3; H)$ for small $H \not\supset K_4$

Problem

Prove or disprove the existence of

(a) *the edge Folkman number* $F_e(K_3, K_3; \overline{P_2 \cup P_3})$*, and*
(b) *the edge Folkman number* $F_e(K_3, K_3; K_1 + C_4)$*.*

$W_5 = K_1 + C_4 \subset J_5 = K_5 - e$, hence
if $F_e(K_3, K_3; W_5)$ exists, then $F_e(K_3, K_3; J_5) \leq F_e(K_3, K_3; W_5)$.

The analogous statement holds for $\overline{P_2 \cup P_3}$,
with an extra condition implied by $\overline{P_2 \cup P_3} \subset W_5$.

Future work:
Study $F_e(K_3, K_3; H)$ for graphs H on at least 6 vertices (beyond B_4),
Computational projects: existence and bounds.

From Edge- to Vertex-Arrowing

Lemma

For $k \geq s \geq 2$, if graph G is H-free, $H \subset K_{k+1}$, and $G \to (K_s, K_k)^e$, then for every vertex $u \in V(G)$ and $s-1$ colors we have $G - u \to (K_k, \cdots, K_k)^v$.

Corollary

For $2 \leq s \leq k$ and graph $H \subset K_{k+1}$, if $F_e(K_s, K_k; H)$ exists, then $F_v^{s-1}(K_k; H)$ also exists and $F_e(K_s, K_k; H) \geq F_v^{s-1}(K_k; H) + 1$.

Special case with $s = 3$ and $H = K_{k+1}$ gives

$F_e(3, k; k+1) > F_v(k, k; k+1)$.

Back to $F_e(3,3;4)$

and how to earn \$100 from RL Graham

The best known bounds:

$$20 \leq F_e(3,3;4) \leq 785.$$

- Upper bound 785 from a modified residue graph via SDP.
- Ronald Graham Challenge for $\$100$ (2012):
  Determine whether $F_e(3,3;4) \leq 100$.

  Conjecture (Exoo, around 2004):
  - $G_{127} \rightarrow (3,3)^e$, moreover
  - Removing 33 vertices from G_{127} (3 indsets of 11) gives a G_{94} which still looks good for arrowing, if so, worth \$100.
- Lower bound: very hard, crawls up slowly 10 (Lin 1972), 16 (PUR 1999), 19 (RX 2007), 20 (Bikov-Nenov 2016).

Graph G_{127}

Hill-Irving 1982, a cool K_4-free graph studied as a Ramsey graph

$$G_{127} = (\mathcal{Z}_{127}, E)$$
$$E = \{(x, y) | x - y = \alpha^3 \pmod{127}\}$$

Exoo conjectured that $G_{127} \rightarrow (3, 3)^e$.

- resists direct backtracking
- resists eigenvalues method
- resists semi-definite programming methods
- resists state-of-the-art 3-SAT solvers
- amazingly rich structure,
  hence perhaps will not resist a proof by hand ...

R · I · T
Department of Computer Science

Some references

- Xiaodong Xu, Meilian Liang, SPR
  On the Nonexistence of Some Generalized Folkman Numbers
  `arXiv 1705.06268`, May 2017

- Xiaodong Xu, Meilian Liang, SPR
  Chromatic Vertex Folkman Numbers
  `arXiv 1612.08136`, December 2016

- Many papers by Bikov, Dudek, Erdős, Folkman, Graham, Li, Lin, Lu, Nenov, Nešetřil, Rödl, Ruciński, Soifer, Xu, and others ...

Thanks for listening!

R · I · T
Department of Computer Science